The EXIT-Characteristic of Softbit-Source Decoders

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***Abstract*— We outline a new technique to compute the EXIT-characteristic of *softbit-source decoders* analytically without extensive histogram measurements. Based on the analytic considerations it is straightforward to derive a compact determination rule for the maximum value of attainable *extrinsic information*. We also show that the area under the EXIT-characteristic grows almost logarithmically with the prediction gain which is utilizable due to the residual redundancy in the source data.**

***Index Terms*— EXIT-characteristic, softbit-source decoding.**

I. Introduction

Modern speech, audio, and video encoders determine source codec parameters which usually exhibit considerable residual redundancy in terms of a non-uniform distribution, in terms of correlation, or in any other possible mutual dependency of parameters. At the receiver of a digital transmission system, *softbit-source decoding* (SBSD) [1] can exploit this natural residual redundancy for error concealment. SBSD can be applied as component decoder in a TURBO-process like *iterative source-channel decoding* (ISCD) [4]–[7] (see Fig. 1).

The convergence of TURBO-processes can be analyzed using EXIT-charts [7]–[10]. EXIT-charts visualize the exchange of *extrinsic information* between the constituent decoders. Each decoder is represented by an EXIT-characteristic, which describes the *extrinsic information transfer* (EXIT) from the *a priori* input $\underline{a}$ to the *extrinsic* output $\underline{e}$ of the decoder. In the original paper [8], S. ten Brink proposed to determine EXIT-characteristics by histogram measurements. The *a priori* input $\underline{a}$ is modelled by a Gaussian process with a specific mean and variance, and the resulting *extrinsic* output values $\underline{e}$ are recorded in histograms. From the histograms for different Gaussian inputs it is straightforward to compute the EXIT-characteristics [8]. Recently, successful attempts have been made to determine the EXIT-characteristic of simple channel codes analytically, e.g., [2], [3].

In this letter, we derive a similar technique for the analytical computation of the EXIT-characteristics of SBSD. The key motivations are: 1) we can skip extensive histogram measurements; 2) we can avoid the crucial design of the histogram limits and resolution which is required in case of SBSD.

Next, we give a brief review of the model proposed in [2], [3]. Afterwards, we modify this model in view of SBSD, and we outline the new computation rule. Finally, we discuss some specific properties of the EXIT-characteristic of SBSD.

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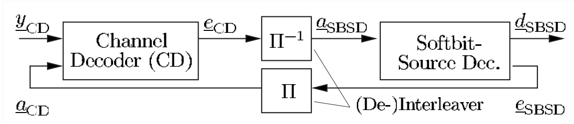

Fig. 1. Exemplary *iterative source-channel decoding* (ISCD) scheme.

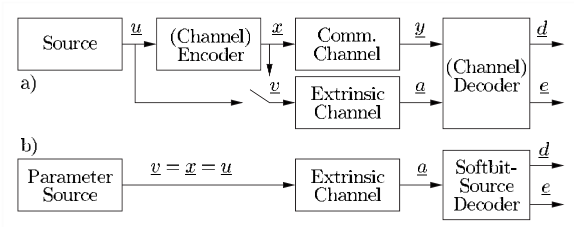

Fig. 2. a) Decoding model proposed in [2], [3]; b) modified model for SBSD.

II. Computation of the EXIT-Characteristic

A. (Channel) Decoding Model as Proposed in [2], [3]

Fig. 2 a) depicts the decoding model which has been proposed by A. Ashikhmin *et al.* in [2], [3]. At time τ a source produces a bit pattern $\underline{u}_\tau$ of N bits. A (channel) *encoder* maps $\underline{u}_\tau$ to a codeword $\underline{x}_\tau$ of length K. After transmission over a *communication channel* the decoder receives a possibly noisy version $\underline{y}_\tau$ for $\underline{x}_\tau$. For convenience, we assume that the single elements of $\underline{y}_\tau$ represent L-values. In addition to $\underline{y}_\tau$ the decoder accepts *a priori* L-values $\underline{a}_\tau$ as soft-inputs. In a TURBO-process, these $\underline{a}_\tau$ are given by the *extrinsic information* provided by the other constituent decoder. Such *extrinsic information* is either available for the coded bits of $\underline{x}_\tau$ (i.e., $\underline{v}_\tau = \underline{x}_\tau$; serial concatenation) or the data bits of $\underline{u}_\tau$ (i.e., $\underline{v}_\tau = \underline{u}_\tau$; parallel concatenation). In either way, the term *extrinsic channel* shall point out that the *extrinsic information* $\underline{a}_\tau$ for $\underline{v}_\tau$ originates from outside the *communication channel*. The evaluation of the soft-input values $\underline{y}_\tau$ and $\underline{a}_\tau$ in consideration of the encoding rule yields the decoder's *a posteriori* L-values $\underline{d}_\tau$ resp. the decoder's *extrinsic* L-values $\underline{e}_\tau$.

B. Modified Decoding Model for SBSD

Softbit-source decoding utilizes natural residual source redundancy [1]. Such residual redundancy can be measured in terms of a non-uniform distribution $P(\underline{u}_\tau)$ of the pattern $\underline{u}_\tau$ or in terms of mutual dependencies in time $P(\underline{u}_\tau|\underline{u}_{\tau-1})$. Thus, in contrast to [2], [3], here the elements $u_\tau^j \in \{+1, -1\}$ of $\underline{u}_\tau$ are usually not independent from each other, and they are usually not equiprobable, i.e. $P(u_\tau^j = \pm 1) \neq 1/2$. The upper index $j = 1, \dots N$ of u_τ^j specifies a particular bit of $\underline{u}_\tau$. In the remainder, the pattern $\underline{u}_\tau$ excluding u_τ^j is written as $\underline{u}_\tau^{[j]}$.

Moreover, the residual redundancy is called to be *natural* because it is implicitly present in the bit pattern $\underline{u}_\tau$. Such residual redundancy has not been artificially introduced by any

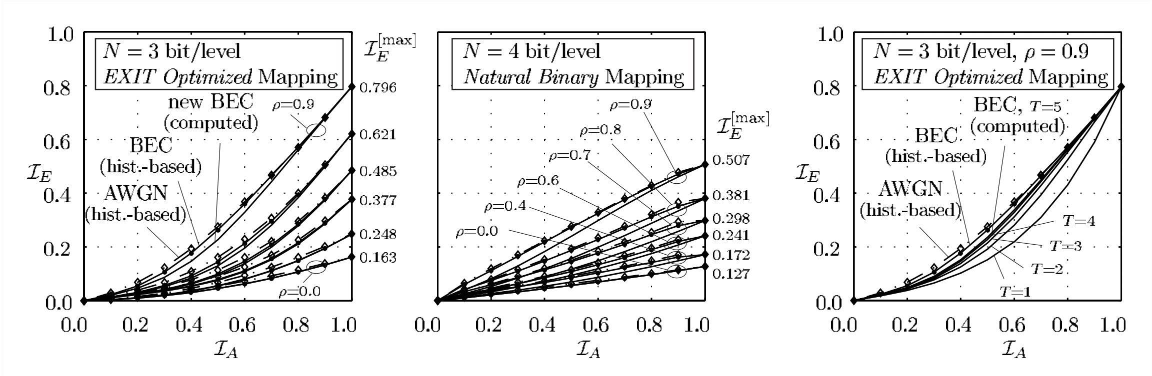

Fig. 3. EXIT-characteristics of SBSD, left/center: two examples for the dependency on ρ and the maximum values for $\mathcal{I}_E^{[\max]}$, right: dependency on T.

specific device. As a consequence, the *encoder* in Fig. 2 a) can be eliminated, and we have $\underline{v}_\tau = \underline{x}_\tau = \underline{u}_\tau$.

Finally, we restrict our considerations to the case that SBSD serves as outer component in a serial concatenation. In this case, SBSD has no access to the L-values $\underline{y}_\tau$ of the *communication channel*. Thus, SBSD is exclusively based on the *a priori* L-values $\underline{a}_\tau$ (see Fig. 2 b)). Notice, the EXIT-characteristics of SBSD for the parallel case can easily be computed from those for the serial case (cmp. to [11]).

C. Histogram-based EXIT-Characteristics of SBSD

Fig. 3 shows some EXIT-characteristics of *softbit-source decoders* which are determined by conventional histogram measurements [6]–[10]. The residual redundancy of the bit patterns $\underline{v}_\tau$ is modeled by a 1st-order Gauss-Markov process with filter coefficient ρ, Lloyd-Max quantization to one out of 2^N levels, and mapping to the pattern $\underline{v}_\tau$ of length N bits. The two leftmost subplots show the results for different values ρ given a fixed value for N and a fixed mapping (left: $N = 3$, *EXIT optimized* [7], center: $N = 4$, *natural binary*).

The dash-dotted curves (diamonds) show simulation results if an *additive white Gaussian noise* (AWGN) channel serves as *extrinsic channel*. The solid curves (bullets) show the corresponding measurements for a *binary erasure channel* (BEC). As a first result we can state that the EXIT-characteristics for both *extrinsic channel* models (AWGN or BEC) are similar. Thus, it is sufficient to derive an analytical determination rule for either of both *extrinsic channels*. In the following, we will focus on the BEC case to keep the computation manageable.

D. New Computation Rule for the BEC Case

The EXIT-characteristic describes the *average extrinsic information* per bit [2], [3], [8]

$$\mathcal{I}_E = \frac{1}{N} \sum_{j=1,\dots N} \mathcal{I}(V^j; E^j) \tag{1}$$

as a function of the *average a priori information* [12]

$$\mathcal{I}_A = (1-q) \cdot \frac{1}{N} \sum_{j=1,\dots N} \mathcal{H}(V^j) \qquad \text{(if BEC).} \tag{2}$$

The term $\mathcal{I}(\cdot;\cdot)$ denotes the *mutual information* and $\mathcal{H}(\cdot)$ the *entropy*. Both terms of information, $\mathcal{I}_A$ and $\mathcal{I}_E$, are coupled via V^j. An upper case letter like V^j marks a random process while the corresponding lower case letter v^j denotes its realization at time τ. The parameter $q \in [0,1]$ in (2) stands for the *erasure probability* of the BEC.

As we consider SBSD to be an outer component in a serially concatenated TURBO scheme, every *extrinsic* L-value e_τ^j is a direct function of present *a priori* L-values $\underline{a}_\tau^{[j]}$ (i.e., $\underline{a}_\tau$ excluding a_τ^j) and past values $\underline{a}_{\tau-1}, \dots \underline{a}_1$ (due to the mutual dependencies in time). Notice, due to the BEC the attainable *a priori* L-values are limited to $a_\tau^j \in \{+\infty, 0, -\infty\}$. We can also write (cmp. [3])

$$\mathcal{I}(V^j; E^j) = \mathcal{I}(V_\tau^j; \underline{A}_\tau^{[j]}, \underline{A}_{\tau-1}, \dots \underline{A}_1) \quad . \tag{3}$$

In order to compute the right hand side of (3) we have to determine the joint probability density function (PDF) $p(v_\tau^j, \underline{a}_\tau^{[j]}, \underline{a}_{\tau-1}, \dots \underline{a}_1)$. Applying the chain rule for probability and assuming a memoryless *extrinsic channel*, enables us to determine the joint PDF recursively,

$$p(v_\tau^j, \underline{a}_\tau^{[j]}, \underline{a}_{\tau-1}, \dots \underline{a}_1) = \sum_{\underline{v}_\tau^{[j]}} p(\underline{a}_\tau^{[j]} | \underline{v}_\tau^{[j]}) \cdot \alpha(\underline{v}_\tau) \tag{4}$$

with

$$\alpha(\underline{v}_t) = \sum_{\underline{v}_{t-1}} P(\underline{v}_t | \underline{v}_{t-1}) \cdot p(\underline{a}_{t-1} | \underline{v}_{t-1}) \cdot \alpha(\underline{v}_{t-1}) \tag{5}$$

for $t = 2, \dots \tau$ and the initialization $\alpha(\underline{v}_1) = P(\underline{v}_1)$. The summation over $\underline{v}_\tau^{[j]}$ resp. $\underline{v}_{t-1}$ means that we have to accumulate over all possible realizations of $\underline{V}_\tau^{[j]}$ resp. $\underline{V}_{t-1}$.

As the residual redundancy in terms of the source statistics $P(\underline{v}_1)$ and $P(\underline{v}_t|\underline{v}_{t-1})$ can be measured once in advance (note, $P(\underline{v}_t|\underline{v}_{t-1})$ is the same for all $t = 2, \dots \tau$ if the Markov process is *homogeneous*) the only terms remaining to be unknown in Eqs. (4),(5) are $p(\underline{a}_\tau^{[j]}|\underline{v}_\tau^{[j]})$ and $p(\underline{a}_{t-1}|\underline{v}_{t-1})$. Both terms can easily be determined for a BEC using

$$p(\underline{a}_t|\underline{v}_t) = (1-q)^{|\underline{a}_t \oplus \underline{v}_t|} \cdot q^{N - |\underline{a}_t \oplus \underline{v}_t|} \tag{6}$$

with the element-wise product and *Hamming weight*

$$|\underline{a}_t \oplus \underline{v}_t| = \sum_{j=1,\dots N} \tanh(a_\tau^j/2) \cdot v_\tau^j \quad . \tag{7}$$

For a given *erasure probability* q the joint PDF (4) can be computed for all possible combinations of v_τ^j, $\underline{a}_\tau^{[j]}, \dots \underline{a}_1$. Thus, we can finally evaluate the *extrinsic information* (3) resp. *average extrinsic information* (1). The bitwise probability $P(v_\tau^j)$, which is required to determine the entropy $\mathcal{H}(V_\tau^j)$ of (2), can be computed as marginal distribution from $P(\underline{v}_\tau)$.

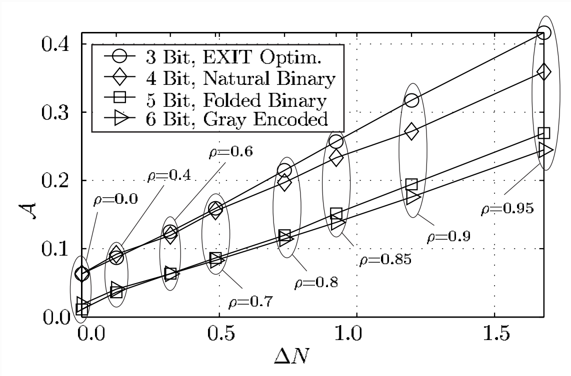

Fig. 4. Area $\mathcal{A}$ under the EXIT-Characteristic of SBSD.

The curves without markers in Fig. 3 depict the EXIT-characteristics if these are computed as described before. For practicability reasons, we limited the history of past *a priori* L-values $\underline{a}_{\tau-1}, \dots \underline{a}_1$ in (4) to the last T time instants, i.e., we computed $\mathcal{I}(V_\tau^j; \underline{A}_\tau^{[j]}, \dots \underline{A}_{\tau-T})$. For correlation $\rho < 0.7$ the computed EXIT-characteristics match very well to the histogram-based curves if $T = 3$. For higher ρ a larger number of T is required[1], e.g., $T = 5$ (see rightmost subplot of Fig. 3).

III. Properties of the EXIT-Characteristic

A. New Computation of Maximum Value for $\mathcal{I}_E$

In [6], [7], [9] a complex technique has been proposed to compute the maximum value for $\mathcal{I}_E$ from the source statistics $P(\underline{v}_1)$ and $P(\underline{v}_t|\underline{v}_{t-1})$. However, the preceding considerations reveal a compact formula which is much easier to compute.

The maximum $\mathcal{I}_E$ will be taken when the *extrinsic channel* (for either model) is error-free. If so, all *a priori* L-values $\underline{a}_\tau$ are scaled versions of $\underline{v}_\tau$. Thus, we can rewrite (1) by

$$\mathcal{I}_E^{[\max]} = \frac{1}{N} \sum_{j=1,\dots N} \mathcal{I}(V_\tau^j; \underline{V}_\tau^{[j]}, \underline{V}_{\tau-1}) \quad . \tag{8}$$

Note, if $\underline{v}_{\tau-1}$ is perfectly known no additional gain can be achieved by the knowledge of the past patterns $\underline{v}_{\tau-2}, \dots \underline{v}_1$. The computed values[2] for $\mathcal{I}_E^{[\max]}$ are also shown in Fig. 3.

B. Area under the EXIT-Characteristic of SBSD

For a proper exchange of *extrinsic information*, in the EXIT-chart the EXIT-characteristic of the 1st decoder must be above the swapped one of the 2nd decoder [2], [3], [8]. The relation implies that the area $\mathcal{A}_{1^{st}}$ under the curve for the 1st decoder must be greater than $1 - \mathcal{A}_{2^{nd}}$ with $\mathcal{A}_{2^{nd}}$ denoting the area under the curve for the 2nd decoder. Thus, in *iterative source-channel decoding* (see Fig. 1) the area $\mathcal{A}_{1^{st}}$ under the curve of channel decoding must exceed the value $1 - \mathcal{A}_{2^{nd}}$ of SBSD.

Fig. 3 reveals that the area $\mathcal{A} = \mathcal{A}_{2^{nd}}$ under the EXIT-characteristic of SBSD depends (at least) on the index assignment, the number N of bits/level, and the correlation ρ. Such correlation can be reduced or even eliminated by advanced source coding like *linear prediction*. The resulting *prediction gain* G_p can be used, e.g., for a reduction of data rate

$$\Delta N = \frac{1}{2} \cdot \mathrm{ld}\, G_p \quad \text{with} \quad G_p = \frac{1}{1-\rho^2} \quad . \tag{9}$$

Fig. 4 shows $\mathcal{A}$ as a function of ΔN for the two leftmost subplots of Fig. 3. Moreover, Fig. 4 depicts the simulation results for some even larger N and some additional bit mappings. As an interesting result we can observe that the area grows almost linearly with ΔN resp. almost logarithmically with G_p. None of the curves starts in the origin because, even if $\rho = 0.0$, SBSD can still exploit the distribution $P(\underline{v}_\tau)$.

IV. Conclusions

We applied recent results to EXIT-charts to the (iterative) joint source-channel decoding problem. EXIT-characteristics of SBSD have been computed analytically which match well to the measured ones for most configurations of correlation, quantizer resolution, and index assignment. We presented a compact formula for the easy computation of the highest possible *extrinsic information* value $\mathcal{I}_E^{[\max]}$, and we demonstrated that the area $\mathcal{A}$ under the EXIT-characteristic grows almost logarithmically with the prediction gain. Maximizing $\mathcal{I}_E^{[\max]}$ or $\mathcal{A}$ might serve as design criterion for future ISCD schemes.

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[1] Note, the computational complexity increases with raising numbers of T. However, computing the EXIT-characteristic is still beneficial because we can avoid the crucial design of histogram parameters which is required by SBSD.

[2] In some cases the values $\mathcal{I}_E^{[\max]}$ differ from those mentioned in [6], [7], [9] because averaging over the bit position j is done in a different way. In Eq. (8) the mean of the N values for $\mathcal{I}(V_\tau^j; \underline{V}_\tau^{[j]}, \underline{V}_{\tau-1})$ is determined. In [6], [7], [9] averaging is done by using a single histogram (instead of N) for all u_τ^j.