# System Identification with Perfect Sequence Excitation – Efficient NLMS vs. Inverse Cyclic Convolution

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## Abstract

Linear transmission systems are often characterized by their impulse responses. A simple and fast approach to acquire these impulse responses is the normalized least-mean-square (NLMS) algorithm in combination with a perfect sequence excitation. It is not only applicable to static impulse response measurements, but has been optimized especially for the tracking of time varying linear systems.

In this paper, different implementation strategies of the perfect sequence excited NLMS algorithm, namely the efficient NLMS and the inverse cyclic convolution algorithm, are discussed and compared in terms of performance, complexity, and applicability. As a main result it is shown that for certain conditions all algorithmic variants can be transferred to each other.

## 1 Introduction

Several approaches have been proposed to solve the problem of system identification and tracking given the input and the output signal of an unknown linear transmission system. A digital model for a typical system identification approach is illustrated in Fig. 1. The unknown linear transmission system and the adaptive filter are represented by their time varying impulse responses in vector notation $\mathbf{g}(k)$ and $\mathbf{h}(k)$, respectively, with time index $k$. In order to consider the influence of an unavoidable measurement noise, $n(k)$ is added to the system response $r(k)$.

One approach to solve this problem is the *normalized least-mean-square* (NLMS) algorithm excited with a *perfect sequence* (PSEQ) $\tilde{p}(k)$ acccording to [1, 2], where the adaptation process is driven by the error signal $e(k)$. This technique has been used for a wide range of applications. On one hand, it can be used with the focus on *identification*, e.g., to perform static measurements, such as measurements of acoustic transfer functions. On the other hand, due to its rapid tracking ability the PSEQ excited NLMS process is also capable to cope with time variant system indentification. In these applications rather the *tracking* aspect is predominant. Examples are simulations of time variant room impulse responses [2], medical applications as sonotubometry for real-time monitoring of the Eustachischen tube activity [3], or continuously measured and tracked *head-related transfer functions* (HRTFs) [4].

In [5, 6], a new *efficient* version of the PSEQ excited NLMS process has been proposed. The advantage is its drastic reduction of the computational effort. Besides its relation to the conventional NLMS algorithm the efficient NLMS also bears a certain resemblance to a simple *inverse cyclic convolution* (ICC) approach. For this reason we want to provide a deeper understanding regarding the different algorithms and their relations. Besides the theoretical and numerical aspects especially complexity properties are of interest for real-time implementations.

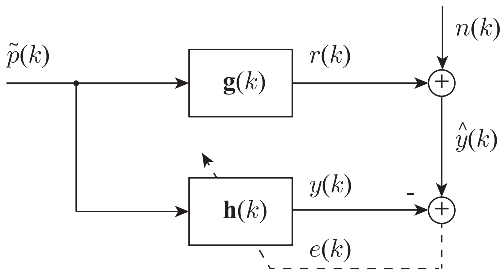


**Figure 1:** Digital model for system identification.

The paper is organized as follows. Sec. 2 summarizes the main characteristics of PSEQs as far as they are of relevance for the theoretical derivations. The objective of Sec. 3 is to give a review of the three considered system identification variants, i.e., the NLMS, the efficient NLMS, and the ICC. Hereafter, the direct mathematical comparison between the algorithms is evolved in Sec. 4. Finally, simulation results in Sec. 5 testify the theoretical conclusions and complexity aspects are presented.

## 2 Perfect Sequences (PSEQs)

All system identification algorithms considered in this paper are excited with so called *perfect sequences* (PSEQs). A PSEQ is a discrete sequence $p(k)$ of length $N$ with an impulse like periodic autocorrelation function

$$\tilde{\varphi}_{pp}(\lambda) = \sum_{k=0}^{N-1} \tilde{p}(k)\tilde{p}(k+\lambda) \tag{1}$$

$$= \begin{cases} \|\mathbf{p}(k)\|^2 & \text{for } \lambda \bmod N = 0 \\ 0 & \text{else}, \end{cases} \tag{2}$$

where $\tilde{p}(k)$ denotes the periodically repeated sequence $p(k)$ and $\|\cdot\|^2$ the quadratic norm. For notational clarity we introduce the PSEQ excitation vectors $\mathbf{p}(k) = (\tilde{p}(k), \tilde{p}(k-1), \ldots, \tilde{p}(k-N+1))^{\mathrm{T}}$. Since $\tilde{p}(k)$ is periodic, only $N$ different excitation vectors

$$\mathbf{p}(k) = \mathbf{p}_j \tag{3}$$

exist with $j = k \bmod N$. As the excitation vectors are cyclically shifted versions of each other, all excitation vectors have the same energy according to

$$E_{\mathbf{p}} = \|\mathbf{p}(k)\|^2 = \|\mathbf{p}_j\|^2 = \mathbf{p}_j^{\mathrm{T}}\mathbf{p}_j \ \text{ with } \ j = k \bmod N. \tag{4}$$

The *discrete Fourier transformation* (DFT) of a PSEQ has a constant magnitude with

$$|\mathrm{DFT}\{\tilde{p}(k)\}_N| = |P(k,\mu)| = 1/\sqrt{E_{\mathbf{p}}}. \tag{5}$$

Furthermore, according to [3] for PSEQs the relation

$$\mathrm{IDFT}\left\{\frac{1}{P(k,\mu)}\right\}_N = \tilde{p}^{(-1)}(k) = \frac{1}{E_\mathbf{p}}\,\tilde{p}(-k) \qquad (6)$$

holds. Combining a constant magnitude spectrum with an arbitrary odd-symmetric phase spectrum according to

$$\phi(0) = 0, \pm\pi \qquad (7)$$
$$\phi(N-\mu) = -\phi(\mu) \quad \text{for } 0<\mu<N \qquad (8)$$

results – after inverse DFT (IDFT) – always in a real-valued, perfect sequence, which can be used to construct different classes of PSEQs (cf. [7]).

# 3 System Identification

Three different algorithms for the identification of $\mathbf{g}(k)$ in terms of $\mathbf{h}(k)$ will be discussed, the NLMS, the efficient NLMS, and the inverse cyclic convolution (ICC). The focus is how the algorithms behave in case of a PSEQ excitation and how they are related to each other.

## 3.1 NLMS

The first approach to be considered is the classical *normalized least-mean-square* (NLMS) algorithm which is driven by the adaption

$$\mathbf{h}(k+1) = \mathbf{h}(k) + \alpha\frac{e(k)\mathbf{p}(k)}{\|\mathbf{p}(k)\|^2} = \mathbf{h}(k) + \alpha\frac{e(k)\mathbf{p}(k)}{E_\mathbf{p}} \qquad (9)$$

with stepsize $\alpha$. In [1] it has been shown that due to the special orthogonal properties of the PSEQs, the NLMS algorithm enables the identification of an unknown *linear time invariant* (LTI) system of dimension $N$ within $N$ iterations.

## 3.2 Geometric Interpretation

To gain a better understanding of the relation between the NLMS and the efficient NLMS from [5, 6], the NLMS adaptation process is geometrically interpreted in terms of a vector space representation as known from literature. For this reason the distance vector $\mathbf{d}(k) = \mathbf{g} - \mathbf{h}(k)$ is introduced as difference between the unknown system vector $\mathbf{g}$ and filter vector $\mathbf{h}(k)$. For the geometric interpretation the unknown system vector $\mathbf{g}$ is assumed to be time invariant, noise signal $n(k)$ is set to zero and stepsize $\alpha = 1$. With these assumptions the NLMS algorithm can be expressed as

$$\begin{aligned}\mathbf{d}(k+1) &= \mathbf{d}(k) - \frac{\mathbf{p}^\mathrm{T}(k)\mathbf{d}(k)}{\|\mathbf{p}(k)\|}\,\frac{\mathbf{p}(k)}{\|\mathbf{p}(k)\|}\\ &= \mathbf{d}(k) - \mathbf{d}^{\|}(k)\,. \end{aligned} \qquad (10)$$

Due to the correlation properties of PSEQs this special class of sequences fulfills the requirements of an *optimal* excitation signal for the NLMS algorithm. An example is illustrated in Fig. 2 for the $N=2$ dimensional vector space with distance vector $\mathbf{d}(k)$ (blue) and the two orthogonal excitation vectors $\mathbf{p}(k)$ and $\mathbf{p}(k+1)$ (red). According to (10) the updated $\mathbf{d}(k+1)$ is obtained by subtracting from $\mathbf{d}(k)$ the orthogonal projection $\mathbf{d}^{\|}(k)$ of $\mathbf{d}(k)$ onto $\mathbf{p}(k)$. Obviously, only the component into the direction of the excitation vector $\mathbf{p}(k)$ contributes to the reduction of the

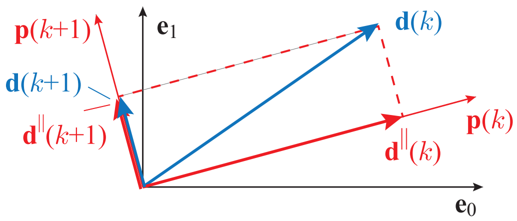


**Figure 2:** Geometric interpretation of NLMS and efficient NLMS, $N=2$.

length of vector $\mathbf{d}(k)$. If $N$ successive excitation vectors are orthogonal in the $N$ dimensional vector space, all $N$ dimensions of distance vector $\mathbf{d}(k)$ are eliminated after $N$ iterations. This requirement is fulfilled by PSEQs. All $N$ dimensions are equally excited and can be compensated accordingly.

## 3.3 Efficient NLMS

In the conventional NLMS algorithm, all coefficients of $\mathbf{h}(k)$ change in each time instant, i.e., (9) has to be calculated for all $N$ coefficients $h_i(k)$ with $0 \le i < N$. The basic idea of the efficient NLMS [5, 6] is to perform, prior to adaptation, a coordinate transformation of the adaptive filter $\mathbf{h}(k)$ to a new orthogonal basis according to

$$\mathbf{h}(k) = \sum_{m=0}^{N-1} c_m(k)\mathbf{w}_m \qquad (11)$$

with

$$\mathbf{w}_m = \mathbf{p}_m/E_\mathbf{p} \qquad \forall\, 0 \le m < N \qquad (12)$$

and (3). The new orthogonal basis $\mathbf{w}_m$ is chosen such that in the first iteration the first dimension of the distance vector $\mathbf{d}(k)$ regarding the new basis is eliminated, whereas all other dimensions remain unchanged. Thus, in one iteration exactly one of the $N$ dimensions of the transformed $\mathbf{h}(k)$ is identified. This leads to a drastic reduction of computational complexity by factor $N$, while, as before, the complete system identification is achieved after $N$ iterations.

Accordingly, in contrast to the conventional NLMS in the efficient NLMS algorithm the coefficients $c_m(k)$ of an equivalent representation of vector $\mathbf{h}(k)$ are adapted during the recursion. When needed, the desired impulse response $\mathbf{h}(k)$ can be determined with (11) or alternatively with

$$\mathbf{h}(k) = \mathrm{IDFT}_N\{\mathrm{DFT}_N\{\mathbf{w}_0\}\cdot\mathrm{DFT}_N\{\mathbf{c}(k)\}\}\,, \qquad (13)$$

as the sum in (11) refers to a cyclic convolution.

For the adaptation of $c_m(k)$ we look for the coeffcients which minimize the instantaneous least-mean-square cost function $J(k) = e^2(k) = (\hat{y}(k) - y(k))^2$ applying the gradient method

$$c_m(k+1) = c_m(k) - \frac{\alpha}{2}\frac{\partial J(k)}{\partial c_m(k)},\; 0 \le m < N\,, \qquad (14)$$

which leads to

$$c_m(k+1) = \begin{cases} c_m(k) + \alpha(\hat{y}(k) - c_m(k)) & \text{for } m = k \bmod N \\ c_m(k) & \text{else} \end{cases}$$

using the special orthogonal properties of PSEQs. See [5, 6] for more details. In each iteration only *one* of the $N$ coefficients $c_m(k+1)$ changes and in case of $\alpha = 1$ the above equation even reduces to a simple assignment operation:

$$c_m(k+1) = \begin{cases} \hat{y}(k) & \text{for } m = k \bmod N \\ c_m(k) & \text{else}. \end{cases} \quad (15)$$

After the initialization phase, i.e., if $k \geq N$, (15) can be expressed in compact notation as

$$c_m(k+1) = \hat{y}(k-\ell) \quad (16)$$

with $m = (k-\ell) \bmod N$ and $0 \leq \ell < N$.

Thus, for a static acoustic measurement of a room impulse response, for instance, we assign with (15) the latest $N$ samples of the measured system response $\hat{y}$ to $c_m$ and perform afterwards a cyclic convolution with (13). This procedure shows a certain analogy to a conventional inverse cyclic convolution, which will be studied within the next sections.

### 3.4 Inverse Cyclic Convolution (ICC)

In acoustics, unknown impulse responses are often measured with a simple ICC approach, i.e.,

$$\mathbf{h}(k) = \mathrm{IDFT}_N \left\{ \frac{\mathrm{DFT}_N\{\hat{\mathbf{y}}(k)\}}{\mathrm{DFT}_N\{\mathbf{p}(k)\}} \right\} \quad (17)$$

or, equivalently, in scalar representation

$$h_i(k+N) = \sum_{m=0}^{N-1} \hat{y}(k+m) \cdot \tilde{p}^{(-1)}(i-(k+m)). \quad (18)$$

In [3] it has been shown, that for $\alpha = 1$ and a PSEQ excitation the NLMS algorithm and the ICC provide the same results within computational accuracy, which is sketched in the following. Starting with (9) and exploiting the orthogonal properties of PSEQs an expression for $\mathbf{h}(k+N)$ according to

$$\mathbf{h}(k+N) = (1-\alpha) \cdot \mathbf{h}(k) + \frac{\alpha}{E_{\mathbf{p}}} \sum_{m=0}^{N-1} \hat{y}(k+m) \cdot \mathbf{p}(k+m)$$

can be derived recursively or in scalar notation

$$h_i(k+N) = (1-\alpha) \cdot h_i(k) + \frac{\alpha}{E_{\mathbf{p}}} \sum_{m=0}^{N-1} \hat{y}(k+m) \cdot \tilde{p}(k+m-i).$$

for all $0 \leq i < N$. For $\alpha = 1$ we get

$$h_i(k+N) = \frac{1}{E_{\mathbf{p}}} \sum_{m=0}^{N-1} \hat{y}(k+m) \cdot \tilde{p}(k+m-i). \quad (19)$$

Using the periodicity of the PSEQ and (6), it turns out that (19), derived from the NLMS algorithm, is equivalent to the convolution of the system response $\hat{y}(k)$ with the inverse excitation signal $\tilde{p}(k)$ as known from (18).

## 4 Equivalence

The direct comparison of (18) and (19) derived from the ICC and NLMS with PSEQ excitation, respectively, proves for $\alpha = 1$ the numerical equivalence of both approaches. In this section it will be shown that also the efficient NLMS algorithm represented by (11) is identical to eqs. (18) and (19), i.e., that all versions provide the same results.

The scalar representation of (11) at time instant $k+N$ is given by

$$h_i(k+N) = \frac{1}{E_{\mathbf{p}}} \sum_{m=0}^{N-1} c_m(k+N) \cdot \tilde{p}(\mu N + m - i) \quad (20)$$

using (12) and an arbitrary $\mu \in \mathbb{Z}$. First, let us note that variable $c_m$ refers to a cyclically shifted representation of $N$ samples of the system response $\hat{y}$ according to (16). Also it is of interest that $\mathbf{w}_m$ in (11) or the periodic sequence $\tilde{p}$ in (20), respectively, is not a function of time instant $k$ in contrast to (19).

In order to find an equivalent expression for $c_m(k+N)$ in (20) we substitute $k+1 = k' + N$ in (16), which leads to

$$c_m(k'+N) = \hat{y}(k'+N-1-\ell) \quad (21)$$

with $m = (k'+N-1-\ell) \bmod N$ and $0 \leq \ell < N$. With a second substitution $\ell' = N-1-\ell$ we find that

$$c_m(k'+N) = \hat{y}(k'+\ell') \quad (22)$$

for $m = (k'+\ell') \bmod N$ and $0 \leq \ell' < N$. For notational simplicity, variables $k' = k$ and $\ell' = \ell$ are replaced again such that we obtain for (20) the expression

$$h_i(k+N) = \frac{1}{E_{\mathbf{p}}} \sum_{\ell=0}^{N-1} c_{(k+\ell) \bmod N}(k+N) \cdot \tilde{p}(\mu N + (k+\ell) \bmod N - i), \quad (23)$$

where the substitution of variable $m$ in the sum of (20) provokes simply a reordering of the summands. As $\tilde{p}(k)$ is periodic with $N$ we find that

$$\begin{aligned} h_i(k+N) &= \frac{1}{E_{\mathbf{p}}} \sum_{\ell=0}^{N-1} \hat{y}(k+\ell) \cdot \tilde{p}(\mu N + (k+\ell) \bmod N - i) \\ &= \frac{1}{E_{\mathbf{p}}} \sum_{\ell=0}^{N-1} \hat{y}(k+\ell) \cdot \tilde{p}(k+\ell-i). \end{aligned} \quad (24)$$

A comparison of equation (24) with (18) and (19) proves that in case of a PSEQ excitation and a stepsize of $\alpha = 1$ the NLMS, the efficient NLMS as well as the ICC provide the same results. For the derivation of (18), (19), and (24) we neither assumed that $n(k) \neq 0$ nor that $\mathbf{g}(k)$ is time invariant. Thus, the equivalence between the three approaches also holds for real measured signals, i.e., in all three cases the noise signal $n(k)$ and the time variance of $\mathbf{g}(k)$ are treated in the same way. Differences only occur in case of a stepsize $0 \leq \alpha < 1$ or during the initialization phase, i.e., when $0 < k < 2N$. These theoretical results will now be verified via simulation.

## 5 Results

The aim of this section is twofold. First, some simulation results are presented to visualize the equivalence between the three adaptation algorithms and to show the influence of the excitation signal itself. And secondly, a short discussion concerning the complexity aspects of the three implementation variants provides some more insight in their algorithmic characteristics.

### 5.1 Simulation

For the comparison, the NLMS, the efficient NLMS algorithm, and the ICC have been considered. For the simulation we chose a system $\mathbf{g}(k)$, which is time invariant despite of a sudden change at $k = 1024$, and an adaptive filter $\mathbf{h}(k)$, both of length $N = 256$. The stepsize is set to $\alpha = 1$

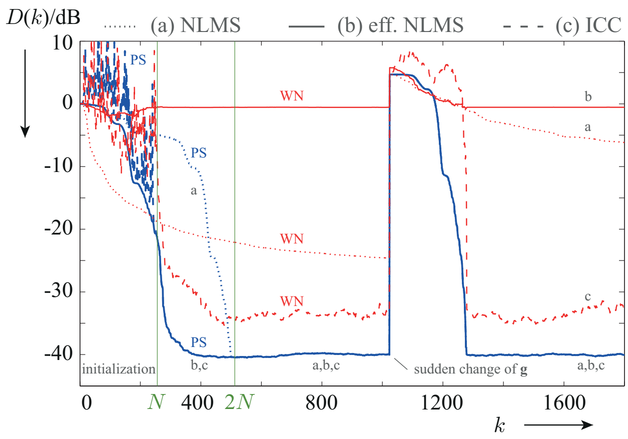


**Figure 3:** Comparison of NLMS, efficient NLMS, and ICC for a perfect sweep (PS) and a periodic white noise (WN) excitation ($N = 256,\ \alpha = 1$).

and we considered a measurement white noise floor $n(k)$ of $-40\,\mathrm{dB}$. As excitation signals we applied a perfect sweep (PS) according to [7] as special PSEQ and in addition, a periodic *white noise* (WN) signal. Fig. 3 summarizes the results in terms of the *system distance*:

$$D(k)/\mathrm{dB} = 10\,\mathrm{lg}(||\mathbf{g}(k) - \mathbf{h}(k)||^2 / ||\mathbf{g}(k)||^2)\,.$$

Inspecting the results of the PSEQ excitation (PS, blue) we can observe differences of all three algorithms during the initialization phase ($0 < k < N$). For $k \geq N$ the efficient NLMS and the ICC approach provide identical system distances. After all filter states are filled ($k \geq N$) the NLMS algorithm takes $N$ iterations for a complete identification. In this phase it works slightly different compared to the other algorithms. After $2N$ iterations, however, the three algorithms provide identical curves, irrespective of a time variance of $\mathbf{g}(k)$ or the occurance of $n(k)$, which confirms the results of Sec. 3.4 and 4.

As expected the curves for a periodic white noise (WN, red) excitation show different progression for each adaptation algorithm and worse results compared to a PSEQ excitation.

## 5.2 Complexity

The complexity balance of the different algorithms strictly depends on the particular application, the corresponding dimensions, and especially the required update rate of $\mathbf{h}(k)$.

It is well known that the NLMS algorithm takes an order of $\mathcal{O}(2N)$ (multiply/add) operations per iteration.

The complexity of the efficient NLMS (for $\alpha = 1$) is mainly determined by (13), which requires for each $\mathbf{h}(k)$ essentially one DFT of order $N$, $N$ complex multiplications, and one IDFT. As $\mathbf{w}_0$ is constant, the corresponding DFT has to be performed only once in an actual implementation.

If the update rate of $\mathbf{h}(k)$ is a multiple of $N$, the ICC approach represented by (17) has basically the same complexity as the efficient NLMS, since the DFT$\{\mathbf{p}(k)\}$ is constant. Otherwise, all different DFT$\{\mathbf{p}_j\}$ for $0 \leq j < N$ can be precalculated and stored efficiently in a lookup table. Furthermore, the complex divisions in (17) can easily be expressed as multiplications in case of a PSEQ excitation.

Let us consider two applications. First, we perform a static measurement of an acoustic impulse response. In this case, we have to take the last $N$ samples of the measured system response $\hat{y}(k)$ according to (15) and determine $\mathbf{h}(k)$ with (13). Alternatively, we can chose the ICC approach leading to the same result with same complexity.

The second example refers to the acquisition of *head-related impulse responses* (HRIRs) [4] continuously measured during a 360° rotation of the object. In practice, it is beneficial to take only every $N^{\mathrm{th}}$ impulse response $\mathbf{h}(k)$. The resolution of impulse responses is still high enough that a simple linear interpolation is sufficient when intermediate HRIRs are needed for auralization or other purposes. On one hand, the NLMS algorithm is used iteratively, where only every $N^{\mathrm{th}}$ impulse response was stored. On the other hand, the efficient NLMS and the ICC approach were performed with an update rate of $N$ time instants. All algorithms provide the same results, however, the efficient NLMS and the ICC approach speed up compuation by factor 70 for $N = 308$ (and factor 400 for $N = 4096$) compared to the NLMS algorithm.

# 6 Conclusions

In this contribution a theoretical analysis of the PSEQ excited NLMS algorithm and related implementation variants was performed for a deeper understanding. Thereby, the efficient NLMS algorithm was proven to be equivalent to a inverse cyclic convolution operation, i.e. the ICC approach. As a result, for a stepsize of $\alpha = 1$ and after the first $2N$ iterations all algorithms show identical performance. Furthermore, in all variants the measurement noise $n(k)$ and a time variance of the unknown linear system have the same impact on the achieved results. For many applications where a low update rate of $\mathbf{h}(k)$ is sufficient, the ICC as well as the efficient NLMS allow for a significant reduction of complexity.

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