International Symposium on Information Theory and its Applications, ISITA2006
Seoul, Korea, October 29–November 1, 2006

On the Optimum Performance Theoretically Attainable for Scalarly Quantized Correlated Sources

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Abstract

Turbo processing enables appropriately designed systems to operate close to their capacity limits. In this contribution we present an upper bound, the optimum performance theoretically attainable (OPTA), that can be achieved when transmitting the samples of scalarly quantized correlated sources. This OPTA limit is based on the combination of the channel capacity and and the distortion rate function. To incorporate the effects of the finite block sizes in real systems, additionally the inevitable loss is taken into account by the sphere-packing bound. In a simulation example we present a multi-mode iterative source-channel decoding scheme that is based on highly redundant index assignments. By its Turbo processing at the receiver and due to its multi-mode flexibility this system can approach the OPTA limit in a wide range of channel conditions.

1. INTRODUCTION

With the discovery of Turbo codes channel coding close to the Shannon limit becomes possible with moderate computational complexity. In the past years the Turbo principle of exchanging *extrinsic* information between separate channel decoders has been adapted to other receiver components. One example is *iterative source-channel decoding* (ISCD) for fixed-length coded sources [1, 2].

The source encoding in today's communication systems is often based on the extraction of (fixed-length coded) parameters such as scale factors or predictor coefficients. These parameters still contain a large amount of residual redundancy. This residual redundancy occurs due to imperfect source encoding, resulting for example from the delay constraints of, e.g., 20 ms in GSM. The *a priori* knowledge on the residual redundancy, e.g., non-uniform probability distribution or auto-correlation, can be utilized by ISCD.

In this paper we present the new capacity limits if the source coding and its residual redundancy is taken into account. Using the distortion rate function, the channel capacity, and the sphere-packing bound, the optimum performance theoretically attainable (OPTA) for the transmission of correlated scalarly quantized sources with a finite block size will be derived.

This work has been supported by the Deutsche Forschungsgemeinschaft DFG.

Thorsten Clevorn has been with the Institute of Communication Systems and Data Processing (IND) RWTH Aachen University, Germany.

2. OPTIMUM PERFORMANCE THEORETICALLY ATTAINABLE (OPTA)

The performance limit for the considered problem is mainly influenced by two factors, the properties of the channel (or transmission) and the properties of the source. The properties of the channel are described by the channel capacity, which gives the maximum information (in bit) that can be transmitted without errors. The properties of the source are described by the distortion rate function, which gives the minimum number of bits required to obtain a certain quality, i.e., stay below a certain distortion. Thus, when the minimum number of bits required by the source equals the maximum number of bits possible on the channel we obtain the performance limit OPTA.

For the derivation we consider exemplarily Gauss-Markov sources which generate zero-mean Gaussian distributed samples u with a variance $\sigma_u^2 = 1$ and an auto-correlation ρ. The samples of the source can resemble, e.g., parameters generated by source encoders for speech, audio, or video signals.

2.1. Channel Capacity

The information rate on the channel is limited by the channel capacity $\mathcal{C}$ as defined by Shannon in [3]. In Fig. 1 the channel capacity per channel use is depicted for a classic AWGN channel. The well known results for Gaussian input and different common signal constellation sets are shown. For details and the channel capacity of other input and channel types and we refer to the literature, e.g., [3–6]. Furthermore, the discussion in the following will be limited to the BPSK case for simplicity, but the extension to other cases is straightforward.

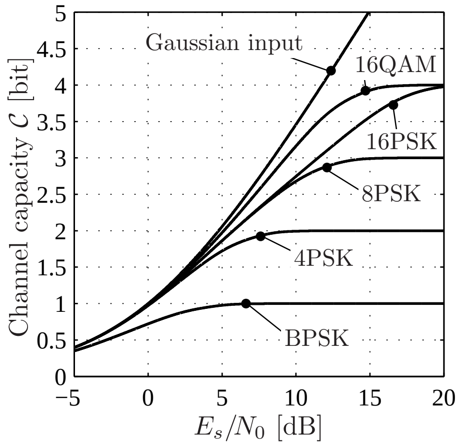

Figure 1: Channel capacity $\mathcal{C}$ per channel use for an AWGN channel and different signal constellation sets.

The channel capacity $\mathcal{C}$ gives the maximum amount of information that can be transmitted over the channel. In the next step the amount of information provided by the source, i.e., generated by source encoding, and the corresponding quality has to be considered. This is taken into account by the distortion rate function discussed in the next section.

2.2. Distortion Rate Function (DRF)

To incorporate the effects of the quantization of the samples u and their auto-correlation ρ, we consider the DRF [7] of the rate distortion theory. The DRF provides a lower bound on the distortion $\mathcal{D}$ for a given number of bits $M = \mathrm{ld}(Q)$ per quantized source sample, and hence an upper bound on the achievable parameter SNR $\mathcal{P}$, the SNR between the original samples (i.e., parameters) u and the estimated values $\hat{u}$ at the receiver. The parameter SNR is a simple but quite accurate and commonly used quality criterion. Note that M represents the (information) "rate" in the DRF.

2.2.1. Uncorrelated sources

For a memoryless (auto-correlation $\rho = 0.0$) zero-mean Gaussian source with variance σ_u^2 and an optimal vector quantizer (OVQ) with infinite size, the DRF is given by [7]

$$\mathcal{D}^{[\mathrm{OVQ}]}_{\rho=0.0}(M) = 2^{-2M}\sigma_u^2 \quad . \tag{1}$$

In the following, we assume for simplicity $\sigma_u^2 = 1$. This DRF has a linear shape in the typical semi-logarithmic plot in Fig. 2. On the right axis the corresponding parameter SNR $\mathcal{P}$ is given,

$$\mathcal{P} = \sigma_u^2/\mathcal{D} = 1/\mathcal{D} \quad . \tag{2}$$

In the case of scalar Lloyd-Max quantization (LMQ) [7] the maximum attainable parameter SNR $\mathcal{P}^{[\max]}(Q)$ for quantization with Q quantization levels $\bar{u}$ is with $\mathrm{E}\{|u|^2\} = 1$

$$\mathcal{P}^{[\max]}_Q = \frac{\mathrm{E}\{|u|^2\}}{\mathrm{E}\{|u-\bar{u}|^2\}} = \frac{1}{1-\mathrm{E}\{|\bar{u}|^2\}} \quad . \tag{3}$$

For non-integer values of Q we assume the transmission with $\lfloor Q \rfloor$ levels in $r_{\lfloor Q \rfloor} = Q - \lfloor Q \rfloor$ of the time instances and with $\lceil Q \rceil$ levels in $r_{\lceil Q \rceil} = \lceil Q \rceil - Q$ of the time instances, $r_{\lfloor Q \rfloor} + r_{\lceil Q \rceil} = 1$. Furthermore, to maximize the parameter SNR we have to unequally distribute the energy to the two cases similar to water-filling. By distribution of the energy we refer to artificially increasing or decreasing the variance (or energy) $\sigma_u^2 = E_u$ of the source signal u to energies $E_u^{[\lfloor Q \rfloor]}$ and $E_u^{[\lceil Q \rceil]}$. An average energy of $\sigma_u^2 = E_u = 1$ yields the side condition

$$r_{\lfloor Q \rfloor} E_u^{[\lfloor Q \rfloor]} + r_{\lceil Q \rceil} E_u^{[\lceil Q \rceil]} = 1 \quad . \tag{4}$$

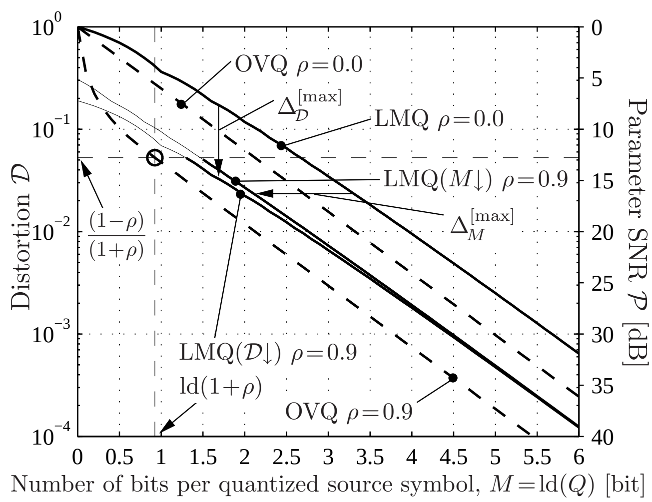

Figure 2: DRF for Gauss-Markov sources with $\rho = 0.0$ and $\rho = 0.9$.

Maximization of (3) yields the maximization of

$$\mathrm{E}\{|\bar{u}|^2\} = r_{\lfloor Q \rfloor} E_u^{[\lfloor Q \rfloor]} \, \mathrm{E}_{\lfloor Q \rfloor}\{|\bar{u}|^2\} + r_{\lceil Q \rceil} E_u^{[\lceil Q \rceil]} \, \mathrm{E}_{\lceil Q \rceil}\{|\bar{u}|^2\}. \tag{5}$$

The values of $\mathrm{E}_{\lfloor Q \rfloor}\{|\bar{u}|^2\}$ and $\mathrm{E}_{\lceil Q \rceil}\{|\bar{u}|^2\}$ (which assume $\sigma_u^2 = E_u = 1$) are scaled by the energies $E_u^{[\lfloor Q \rfloor]}$ and $E_u^{[\lceil Q \rceil]}$. Using (3) to replace the $\mathrm{E}_{(\cdot)}\{|\bar{u}|^2\}$ by the respective $\mathcal{P}_{(\cdot)}^{[\max]}$, the maximization of (5) is solved by

$$E_u^{[\lfloor Q \rfloor]} = -\frac{r_{\lfloor Q \rfloor} \mathcal{P}_{\lceil Q \rceil}^{[\max]}}{\Upsilon} \pm \sqrt{\left(\frac{r_{\lfloor Q \rfloor} \mathcal{P}_{\lceil Q \rceil}^{[\max]}}{\Upsilon}\right)^2 + \frac{\mathcal{P}_{\lceil Q \rceil}^{[\max]}}{\Upsilon}}, \tag{6}$$

$$\text{with} \quad \Upsilon = (1 - r_{\lfloor Q \rfloor}^2) \mathcal{P}_{\lfloor Q \rfloor}^{[\max]} - r_{\lfloor Q \rfloor}^2 \mathcal{P}_{\lceil Q \rceil}^{[\max]} \quad ,$$

and using the solution of (6) that fulfills

$$0 \leq E_u^{[\lceil Q \rceil]} \leq 1 \leq E_u^{[\lfloor Q \rfloor]} \leq 1/r_{\lfloor Q \rfloor} \quad . \tag{7}$$

Since the qualtity in terms of parameter SNR and not the fixed throughput is the target of the optimization, the energy for the bad channel, i.e., lower parameter SNR $\mathcal{P}^{[\max]}$, has to be increased, $E_u^{[\lfloor Q \rfloor]} \geq 1$.

With this method we obtain the uppermost curve (LMQ $\rho = 0.0$) in Fig. 2. An impairment compared to the OVQ can be observed which is constant for higher values of M.

2.2.2. Correlated sources

Sources with memory, e.g., an auto-correlation $\rho > 0$, permit greater data compression than memoryless sources for a given $\mathcal{D}$ [7].

For an OVQ and small distortions

$$\mathcal{D}_\rho^{[\mathrm{OVQ}]} \leq \frac{1-\rho}{1+\rho} \quad , \tag{8}$$

the DRF is described by [7]

$$\mathcal{D}_\rho^{[\mathrm{OVQ}]}(M) = (1-\rho^2) \mathcal{D}_{\rho=0.0}^{[\mathrm{OVQ}]}(M) = (1-\rho^2) 2^{-2M} \sigma_u^2 \quad . \tag{9}$$

The term $(1-\rho^2)$ is also known as spectral flatness [7]. The region of small distortions corresponds to rates $M \geq \mathrm{ld}(1+\rho)$.

For distortions $\mathcal{D}$ larger than the limit of (8) the relation between distortion $\mathcal{D}$ and rate M is given parametrically for a zero-mean Gaussian source by [7]

$$\mathcal{D}_\rho^{[\mathrm{OVQ}]}(\Phi) \Bigg|_{\mathcal{D} > \frac{1-\rho}{1+\rho}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \min\{\Phi, S_{uu}(e^{\jmath\omega})\} \, \mathrm{d}\omega \tag{10}$$

$$M_\rho^{[\mathrm{OVQ}]}(\Phi) \Bigg|_{M < \mathrm{ld}(1+\rho)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \max\left\{0, \frac{1}{2} \, \mathrm{ld} \, \frac{S_{uu}(e^{\jmath\omega})}{\Phi}\right\} \mathrm{d}\omega, \tag{11}$$

with the power spectral density (PSD) $S_{uu}(e^{\jmath\omega})$ for an auto-correlation ρ defined by

$$S_{uu}(e^{\jmath\omega}) = \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos\omega} \sigma_u^2 \quad . \tag{12}$$

The DRF for an OVQ and $\rho = 0.9$ is depicted in Fig. 2 too.

In the case of an LMQ there exist two different ways (or a combination of them) to explore the redundancy caused by an auto-correlation $\rho > 0$.

On the one hand, the redundancy can be used to decrease the distortion $\mathcal{D}$ for a certain rate M by linear prediction before the LMQ. We denote this approach by LMQ($\mathcal{D}\downarrow$). The variance of the source signal u can be reduced to $(1-\rho^2)\sigma_u^2$, leading to reduction of the distortion $\mathcal{D}$ by the same factor. This corresponds to a constant vertical offset in Fig. 2.

On the other hand, the rate M can be reduced by [7]

$$\Delta_M = \frac{1}{2} \, \mathrm{ld} \, \frac{1}{1-\rho^2} \tag{13}$$

at a constant distortion $\mathcal{D}$. This could be achieved by an appropriate entropy source encoder after the LMQ and yields a horizontal offset in Fig. 2.

In the following we use the latter approach denoted as LMQ($M\downarrow$) because it only affects the digital, i.e., quantized values, and does not incorporate any processing before quantization. Furthermore, we assume that the curves are only valid in the small distortions regime as indicated by the thinner lines for larger distortions in Fig. 2.

2.3. Performance Limits

Combining the DRF with the well known channel capacity yields the optimum performance theoretically attainable (OPTA). This OPTA limit is an upper bound on the parameter SNR for a certain transmission scenario. In the examples presented in the following we assume that $M^\star = 6$ bits per source sample are transmitted on the channel. This bit rate is distributed to the quantization with $M = \mathrm{ld}(Q)$ bit and channel coding with rate r_C, $M^\star = M/r_\mathsf{C}$.

In Fig. 3 the OPTA limits for the DRFs in Fig. 2 are depicted assuming BPSK transmission over an AWGN channel. BPSK transmission ($I = 1$ bit per channel symbol) of $M^\star = 6$ bits requires $M^\star/I = 6$ channel uses. With $M^\star/I = 6$ channel uses available, the combined channel capacity at a certain E_s/N_0 is $(M^\star/I)\mathcal{C} = 6\mathcal{C}$, with the respective BPSK capacity $\mathcal{C}$. Thus, we can assign at maximum $M = 6\mathcal{C}$ bit per source sample, or in general

$$M = \frac{M^\star}{I} \mathcal{C} \quad , \tag{14}$$

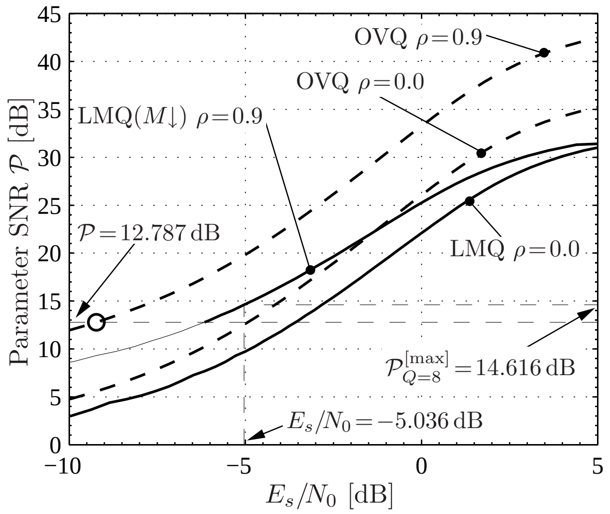

Figure 3: OPTA for $\rho = \{0.0, 0.9\}$, OVQ or LMQ, and $M^\star = 6$ bits/sample.

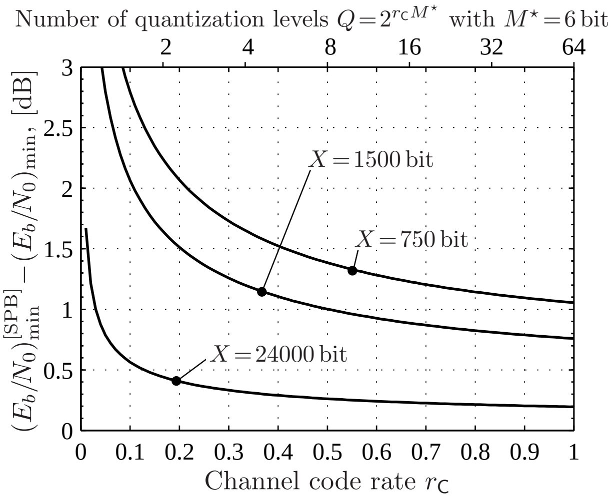

Figure 4: Loss in E_b/N_0 defined by the SPB for different code rates r_C.

when error-free transmission is desired. This relates via the DRF to a minimum distortion $\mathcal{D}$, i.e., a maximum parameter SNR $\mathcal{P}^{[\max]}$. The OPTA limit describes the dependency between this $\mathcal{P}^{[\max]}$ and the channel quality E_s/N_0. It can be observed that the auto-correlation $\rho = 0.9$ increases the OPTA limit and using a LMQ instead of an OVQ yields a noticeable impairment. Note, for LMQ and $\rho = 0.9$ the OPTA limit is exact only in the small distortions regime ($\mathcal{P} \geq 12.787$ dB). Nevertheless, the depicted OPTA curve LMQ($M\downarrow$) gives a loose upper bound for large distortions, i.e., low parameter SNRs.

2.4. Sphere-Packing Bound

So far infinite information and code block sizes are assumed. The effects of finite block sizes are incorporated in the sphere-packing bound (SPB) by Shannon [8]. A detailed discussion can be found in [9]. The SPB assumes a continuous input alphabet. For a binary input channel the random coding (upper) bound by Gallager [9, 10] could be used. However, in [9] it is shown that the differences are negligible for large information block sizes V, $V \gtrsim 100$, and we restrict the considerations to the simpler SPB. The information block size V is in this case the number bits required at minimum for the quantization of the U source codec parameters in a frame with Q levels, $V = U\,\mathrm{ld}(Q) = UM$.

The SPB defines a minimum channel quality $(E_b/N_0)^{[\mathrm{SPB}]}_{\min}$ which is required for error-free transmission with a information block sizes V at a certain code rate r_C. The limit $(E_b/N_0)_{\min}$ for an infinite block size is often referred to as Shannon limit and is a lower bound for $(E_b/N_0)^{[\mathrm{SPB}]}_{\min}$. The difference $(E_b/N_0)^{[\mathrm{SPB}]}_{\min} - (E_b/N_0)_{\min}$ then gives the inevitable impairment for a certain finite block size. In Fig. 4 this difference is plotted versus the code rate r_C for some arbitrary fixed code block sizes $X = V/r_\mathsf{C}$. As well known, the loss in E_b/N_0 decreases for an increasing block size. With a constant code block size X a smaller code rate r_C corresponds to a smaller information block size V. On the top axis the number of quantization levels $Q = 2^{r_\mathsf{C} M^\star}$ are given when $M^\star = 6$ bits per source sample are transmitted after channel encoding with rate r_C.

When the SPB is applied to the OPTA limit derived above (see Fig. 3), we denote the emerging bound by OPTA-SPB.

3. SIMULATION EXAMPLE

In the following we will briefly present a multi-mode iterative source-channel decoding (ISCD) scheme [11, 12] which closely approaches the derived OPTA-SPB limit in a wide range of channel conditions. The baseband model for ISCD in general [1, 2, 13] is depicted in Fig. 5.

The source codec parameters u of a frame $\underline{u}$ are quantized to quantizer levels ξ. These quantizer levels are mapped by the index assignment (IA) to bit patterns $\mathbf{x}$. The complete frame of bit patterns $\underline{\mathbf{x}}$ is permuted by a bit interleaver π to $\underline{\tilde{\mathbf{x}}}$. Afterwards, channel encoding is applied, yielding in this case bipolar bits $\ddot{y}$ for a BPSK transmission.

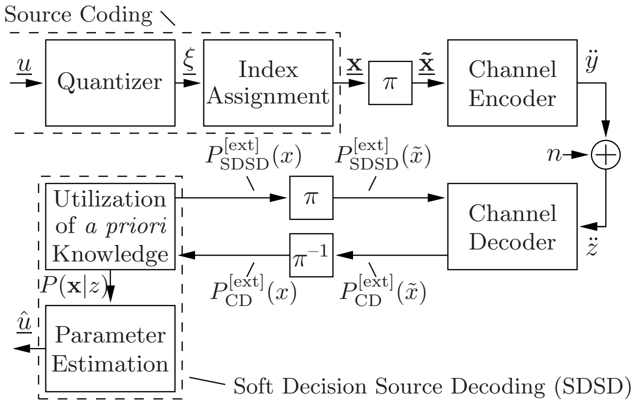

Figure 5: Baseband model of iterative source-channel decoding.

The receiver features a Turbo process which exchanges extrinsic information between the channel decoder and so-called soft decision source decoding (SDSD). The latter one exploits the residual redundancy, e.g., the auto-correlation ρ, in the source samples. In the example of Fig. 6 we assume $\rho = 0.9$. Such high values of auto-correlation still occur for certain source codec parameters in today's communication systems such as GSM or UMTS. For further details on the ISCD algorithm we refer to the literature, e.g., [1, 2, 13] and especially to [11, 12] for the considered multi-mode scheme.

A crucial design element for ISCD is the IA. In [11] a powerful and flexible multi-mode ISCD (MM-ISCD) scheme is presented, which is based on highly redundant IAs. With these IAs the relation between quantizer levels Q and assigned bits M is flexible. In the example, $M = M^\star = 6$ bits are assigned to $Q = \{2, \dots 32\}$ levels by versatile block code (BC) based IAs [12, 14] and a rate $r_\mathsf{C} = 1$ recursive convolutional code with generator polynomial $\mathbf{G} = (\frac{1}{1+D+D^2+D^3})$ is used. Instead of using any specific speech, audio, or video encoder, we model $U = 250$ statistically independent source codec parameters u by U independent Gauss-Markov processes with $\sigma_u^2 = 1$. Each Gauss-Markov process has an auto-correlation of $\rho = 0.9$. The $U = 250$ source codec parameters at one time instance form a frame $\underline{u}$. Thus, the code block size is $X = UM^\star = 1500$.

As visible in Fig. 6 the MM-ISCD system performs very close to the OPTA-SPB limit over a wide range of channel conditions if the optimum number of quantizer levels is chosen at the transmitter. Also depicted is the performance of a classic, non-flexible ISCD system, with $r_\mathsf{C} = 1/2$ channel coding and an EXIT optimized (EO) [13] IA from $Q = 8$ levels to $M = 3$ bits. This scheme comes close to the OPTA-SPB limit only in a small E_s/N_0 region.

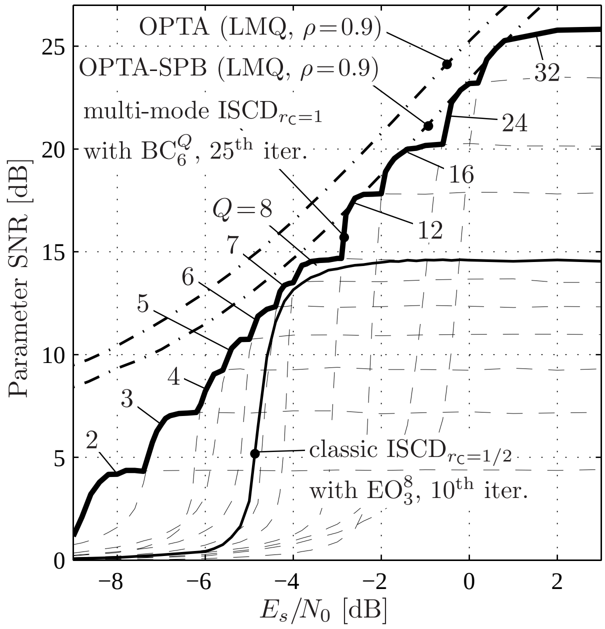

Figure 6: Results for MM-ISCD for $\rho = 0.9$, $X = 1500$, AWGN, BPSK.

When taking a close look, the MM-ISCD system actually exceeds the OPTA-SPB limit in a few cases. This can be explained by the exploitation of the correlation between adjacent frames in the SDSD. Using the knowledge from previous time instances somewhat increases the effective frame (or block) size. The OPTA limit considering only an infinite information block size is an upper bound in any case.

4. CONCLUSIONS

In this paper we presented a bound on the optimum performance theoretically attainable (OPTA) for the transmission of scalarly quantized correlated sources. This OPTA limit is based on the interaction of the channel capacity with the respective distortion rate functions we derived. Furthermore, we additionally incorporated the effects of a finite block size by applying the sphere-packing bound (SPB). With a simulation example of a novel, versatile multi-mode ISCD system we demonstrated that the OPTA-SPB limit can be approached for a wide range of channel conditions with a carefully designed system.

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