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OPTIMALITY OF SEQUENTIAL QUANTIZATION IN ANALYSIS-BY-SYNTHESIS SPEECH CODECS

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ABSTRACT

In analysis-by-synthesis coders the problem of approximating the original signal by the synthesized signal is solved over a limited time interval only. In this contribution a systematic investigation of possible improvements by using extended or even unlimited intervals is presented.

Keywords: Analysis-by-synthesis; vector quantization; LPC; distortion minimization.

1. INTRODUCTION

In linear predictive coding using analysis-by-synthesis techniques [1, 2] the quantization error at the output of a synthesis filter $\frac{1}{P(z)} \cdot \frac{1}{A_w(z)}$ shall be minimized as depicted in Figure 1. The signal is sequentially processed in block mode. As usually done in linear predictive coding schemes the original signal $s(k)$ is modelled as the output of a long term filter $\frac{1}{P(z)}$ and a short term synthesis filter $\frac{1}{A_w(z)}$ excited with the corresponding prediction error or residual sequence $r(k)$. The process $r(k)$ is then approximately white noise. To achieve noise shaping we choose e. g. $A_w(z) = A(z/\alpha)$ with $A(z)$ representing the normal analysis filter.

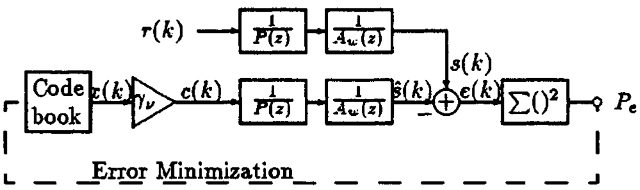

Figure 1: Basic structure of analysis-by-synthesis coders

For each whole block or interval I_ν consisting of L samples the quantized excitation sequence $c_\nu(k)$ is determined, so as to minimize the perceptual quantization error P_e. In the state-of-the-art schemes the error is computed just over the same time interval I_ν (analysis interval), ignoring the influence of the quantization on future intervals $I_{\nu+1}, I_{\nu+2}$ etc. In [3] this influence is taken into consideration to a certain extent only, as the effect of excitation $c_\nu(k)$ on one future frame $I_{\nu+1}$ is considered, but by using an error P_e independent of $c_{\nu+1}(k)$.

In this paper a joint optimization technique is proposed, taking into account the contribution of the excitation sequences $c_\nu(k), c_{\nu+1}(k), \ldots$ to the expected distortion computed over all future frames. A theoretical solution for the described problem is derived and applied to a fixed grid Regular Pulse Excitation codec (RPE, e. g. [2]). With this new systematic analysis, the influence of analysis time limitation in the traditional schemes can be evaluated.

2. SEQUENTIAL QUANTIZATION

The synthesis filter cascade in interval I_ν is described by its impulse responses $h_\nu(k)$ which can be truncated to $R < L$ samples. At first we consider the case that an FIR synthesis filter is used or that the coefficients of an IIR filter have not changed from $I_{\nu-1}$ to I_ν i. e. $h_{\nu-1} = h_\nu$. The synthesized signal in frame I_ν is denoted by $\hat{s}_\nu(k)$ and the input by $c_\nu(k)$. Then the vector $\hat{\mathbf{s}}_\nu = (\hat{s}_\nu(1), \ldots, \hat{s}_\nu(L))^T \in \mathbf{R}^L$ is given by

$$\hat{\mathbf{s}}_\nu = \begin{pmatrix} \hat{s}_\nu(1) \\ \hat{s}_\nu(2) \\ \vdots \\ \vdots \\ \hat{s}_\nu(L) \end{pmatrix} = \Big(\underbrace{\begin{matrix} 0 \cdots 0 & h_\nu(R\!-\!1) \cdots \\ \vdots & \ddots \quad \ddots \\ \vdots & 0 \\ \vdots & \\ 0 \cdots\cdots & \cdots \end{matrix}}_{\mathbf{H}_\nu^1} \Big| \underbrace{\begin{matrix} h_\nu(0) & 0 & \cdots & \cdots & 0 \\ \vdots & h_\nu(0) & 0 & & \vdots \\ h_\nu(R\!-\!1) & & \ddots & \ddots & \vdots \\ \ddots & \ddots & & \ddots & 0 \\ \cdots & 0 & h_\nu(R\!-\!1) & \cdots & h_\nu(0) \end{matrix}}_{\mathbf{H}_\nu^0} \Big) \begin{pmatrix} c_{\nu-1}(1) \\ c_{\nu-1}(2) \\ \vdots \\ c_{\nu-1}(L) \\ c_\nu(1) \\ c_\nu(2) \\ \vdots \\ c_\nu(L) \end{pmatrix} \quad (1)$$

$$= \mathbf{H}_\nu^1 \mathbf{c}_{\nu-1} + \mathbf{H}_\nu^0 \mathbf{c}_\nu$$

In the general case the matrices need not consist of time invariant shifted versions of the same impulse response, but all effects of time variation of the synthesis filter can be covered by proper specification of $\mathbf{H}_\nu^1$ and $\mathbf{H}_\nu^0$. The target vector $\mathbf{s}_\nu$ is equivalently substituted by $\mathbf{s}_\nu = \mathbf{H}_\nu^1\, \mathbf{r}_{\nu-1} + \mathbf{H}_\nu^0 \mathbf{r}_\nu$ with $r_\nu(k)$ denoting the residual sequence.

The speech signal begins in frame I_1, thus $\mathbf{r}_0 = \mathbf{c}_0 = \mathbf{0}$. While M may be arbitrarily large, we obtain P_e summed over M frames as

$$\begin{aligned} P_e &= \sum_{i=1}^{M} D_i^2 = \sum_{i=1}^{M} \|\mathbf{s}_i - \hat{\mathbf{s}}_i\|^2 \\ &= \sum_{i=1}^{M} \left\|\mathbf{H}_i^1(\mathbf{r}_{i-1} - \mathbf{c}_{i-1}) + \mathbf{H}_i^0(\mathbf{r}_i - \mathbf{c}_i)\right\|^2 . \end{aligned} \quad (2)$$

Thus P_e depends on the signal which is only described statistically. The task of an optimum sequential quantization algorithm is now to minimize the expected value of P_e by choosing the optimal excitation signal $\mathbf{c}_\nu$ in each frame.

$$E\{P_e\} = E\left\{\sum_{i=1}^{M} \|\mathbf{s}_i - \hat{\mathbf{s}}_i\|^2\right\} \rightarrow \text{Min} \quad (3)$$

3. MATHEMATICAL SOLUTION

3.1 Problem Description

Since the coder works causally in frame I_ν all previously determined excitation vectors $\mathbf{c}_i$ $(i \leq \nu - 1)$ and the target signals represented by $\mathbf{r}_i$ $(i \leq \nu - 1)$ are given. Normally $\mathbf{r}_\nu$ is known, and due to the often existing delay of LPC-coders we postulate the interesting case that $\mathbf{r}_{\nu+1}$ is given as well.

The quantization in frame I_ν can then be expressed by a function or rule $\mathbf{p}_\nu$ with $\mathbf{c}_\nu = \mathbf{p}_\nu(\mathbf{r}_1, \ldots, \mathbf{r}_{\nu+1}, \mathbf{c}_1, \ldots, \mathbf{c}_{\nu-1})$.

When selecting $\mathbf{c}_\nu$ the distances $D_1, \ldots, D_{\nu-1}$ are already fixed and only $D_\nu, \ldots, D_M$ can be influenced. Previous vectors $\mathbf{r}_i, \mathbf{c}_i$ $(i \leq \nu - 2)$ do not affect the latter, due to the limited length of the impulse response.

Therefore the function $\mathbf{p}_\nu$ has only three arguments, i. e. the task is now to find a set of rules

$$\mathbf{c}_\nu = \mathbf{p}_\nu(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}, \mathbf{r}_\nu, \mathbf{r}_{\nu+1}) \quad \text{for} \quad \nu = 1, \ldots, M \quad (4)$$

that minimizes $E\{P_e\}$.

According to the repeating dependence (4) one should note that $\mathbf{c}_M$ indirectly depends on the selection of $\mathbf{c}_1$. Further, P_e is a function of M random variables $\mathbf{r}_i$ and M functions $\mathbf{p}_i$

$$P_e = P_e\{\mathbf{r}_1, \ldots, \mathbf{r}_\nu, \mathbf{p}_1, \ldots, \mathbf{p}_M\} . \quad (5)$$

The expectation value means averaging over $\mathbf{r}_1, \ldots, \mathbf{r}_M$ thus the problem is

$$E\{P_e\} = f\{\mathbf{p}_1, \ldots, \mathbf{p}_M\} \rightarrow \text{Min} . \quad (6)$$

The described problem is of the class treated by the calculus of variations. Following this theory e. g. [4, pp. 147], the existence of a solution is generally not guaranteed. Usually necessary conditions are imposed first because the question of existence is more complicated.

3.2 Derivation of a Necessary Condition

We now presuppose that the $\mathbf{c}_i$ are the function values of the functions $\mathbf{p}_i$ as given by (4). To derive a condition for $\mathbf{p}_\nu$ the following splitting is advantageous

$$\begin{aligned} &E\{P_e\} \\ &= E\left\{\sum_{i=1}^{\nu-1} D_i^2\right\} + E\left\{\sum_{i=\nu}^{M} \left\|\mathbf{H}_i^1(\mathbf{r}_{i-1} - \mathbf{c}_{i-1}) + \mathbf{H}_i^0(\mathbf{r}_i - \mathbf{c}_i)\right\|^2\right\} . \end{aligned} \quad (7)$$

Since the left hand side expression is not involved by variation of $\mathbf{p}_\nu$ it is equivalent to minimize the right hand expression. To extract the dependence on the function value $\mathbf{c}_\nu$ we assume first that the arguments $\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}, \mathbf{r}_\nu, \mathbf{r}_{\nu+1}$ of $\mathbf{p}_\nu$ see (4) are given (and thus $\mathbf{c}_\nu$) and average afterwards over these arguments as represented by

$$\begin{aligned} &= E\left\{\sum_{i=1}^{\nu-1} D_i^2\right\} + \\ &E\Bigg\{\underbrace{E\left\{\sum_{i=\nu}^{M} \left\|\mathbf{H}_i^1(\mathbf{r}_{i-1} - \mathbf{c}_{i-1}) + \mathbf{H}_i^0(\mathbf{r}_i - \mathbf{c}_i)\right\|^2 \middle| \mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}, \mathbf{r}_\nu, \mathbf{r}_{\nu+1}\right\}}_{\delta_\nu(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}, \mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}, \mathbf{p}_{\nu+1}, \ldots, \mathbf{p}_M)}\Bigg\} \end{aligned} \quad (8)$$

Analyzing the inner expected value, it can be seen that $\mathbf{r}_\nu$ has only influence via $\mathbf{r}_\nu - \mathbf{p}_\nu(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}, \mathbf{r}_\nu, \mathbf{r}_{\nu+1}) = \mathbf{r}_\nu - \mathbf{c}_\nu$ as noted in the abbreviation δ_ν.

Minimizing the inner expected value minimizes the outer and vice versa, therefore M minimization conditions result:

Find a set of functions $\mathbf{p}_i$, with $i = 1, \ldots, M$ such that all $\delta_\nu(\cdots)$ $\nu = 1, \ldots, M$ are minimized.

By varying over the functions $p_{\nu+1}, \cdots, p_M$ and selecting $\mathbf{c}_\nu$ the minimum of δ_ν must be achieved.

A set of solution functions shall be $\varphi_1, \ldots, \varphi_M$ with $\mathbf{c}_\nu^{opt} = \varphi_\nu(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}, \mathbf{r}_\nu, \mathbf{r}_{\nu+1})$. Using a recursion $\delta_\nu = f(\delta_{\nu+1})$ and the minimum conditions, a necessary condition for the φ_ν can be derived as proven in the appendix. The result is stated below:

If $A(\mathbf{c}_\nu, \mathbf{c}_\nu^{opt})$ is defined as

$$\begin{aligned} &A(\mathbf{c}_\nu, \mathbf{c}_\nu^{opt}) = \\ &\left\|\mathbf{H}_\nu^1(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}) + \mathbf{H}_\nu^0(\mathbf{r}_\nu - \mathbf{c}_\nu)\right\|^2 + \\ &\Big\| \mathbf{H}_{\nu+1}^1(\mathbf{r}_\nu - \mathbf{c}_\nu) + \mathbf{H}_{\nu+1}^0\Big(\mathbf{r}_{\nu+1} - \\ &\qquad E\{\varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu^{opt}, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}) | \mathbf{r}_\nu - \mathbf{c}_\nu^{opt}, \mathbf{r}_{\nu+1}\}\Big)\Big\|^2 \end{aligned} \quad (9)$$

with the given function $\varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2})$ then the following relation

$$A(\mathbf{c}_\nu, \mathbf{c}_\nu^{opt}) \geq A(\mathbf{c}_\nu^{opt}, \mathbf{c}_\nu^{opt}) \quad \text{holds for all } \mathbf{c}_\nu \in \mathbf{R}^L. \quad (10)$$

This is an implicit criterion for $\mathbf{c}_\nu^{opt}$. Inserting a test vector $\tilde{\mathbf{c}}_\nu^{opt}$ leads to $A(\mathbf{c}_\nu, \tilde{\mathbf{c}}_\nu^{opt})$. If the minimum of this function is taken for $\mathbf{c}_\nu = \tilde{\mathbf{c}}_\nu^{opt}$, then the necessary condition for $\tilde{\mathbf{c}}_\nu^{opt}$ being an optimal value is fulfilled. With this criterion the problem can generally be solved; at least theoretically. Beginning with $\varphi_M(\cdots)$ and ending with $\varphi_1(\cdots)$ all functions can be obtained recursively. Practically, approximations were found that allow a forward directed proceeding.

If the filter matrices are interpreted as random values, too and are provided up to the same time index as the residual vectors, the expected values must include averaging over the filter matrices. A similar criterion can be derived in this case but its usefulness is very limited because for an application (e. g. RPE) expected values of matrix vector products including inverse filter matrices have to be taken which denies any practical solution.

Instead, by using condition (10) analytical and practical solutions were derived.

4. APPLICATION

For the fixed grid RPE-approach, an analytical solution can be obtained. For this type of quantization the excitation vector must satisfy $\mathbf{c}_\nu = \mathbf{Q}\xi_\nu$ with $\xi_\nu \in \mathbf{R}^k$ and $k < L$, denoting by $\mathbf{Q}$ an $(L \times k)$ position matrix . Since $k < L$ this corresponds to a projection which results in a linear quantization rule. By solving the minimization conditions backward, it can be shown that the quantization functions are given by

$$\mathbf{p}_i(\mathbf{r}_{i-1}-\mathbf{c}_{i-1},\mathbf{r}_i,\mathbf{r}_{i+1}) = \mathbf{Q}(\mathbf{L}_i)^{-1}\mathbf{Q}^T\{\mathbf{A}_i(\mathbf{r}_{i-1}-\mathbf{c}_{i-1}) + \mathbf{B}_i\mathbf{r}_i + \mathbf{D}_i\mathbf{r}_{i+1}\} . \tag{11}$$

with matrices $\mathbf{L}_i, \mathbf{A}_i, \mathbf{B}_i$ and $\mathbf{D}_i$ to be determined recursively. Applying (10) leads to a matrix recursion for all matrices. For the main matrix $\mathbf{L}_i$ it is of the form $\mathbf{L}_i = f\left[(\mathbf{L}_{i+1})^{-1}\right]$ beginning the backward recursion with

$$\mathbf{L}_M = \mathbf{Q}^T(\mathbf{H}^0_M)^T\mathbf{H}^0_M\mathbf{Q} . \tag{12}$$

In practice it turns out that the iteration converges very rapidly, or in other words that the dependence on the initial value $\mathbf{L}_M$ decays very quickly and vanishes for $M \to \infty$. This means that for an almost precise computation of $\mathbf{L}_\nu$ only few future filter matrices $\mathbf{H}^j_\nu, \mathbf{H}^j_{\nu+1}, \cdots$ have to be known.

The application of the theoretic results to a special speech codec based on the GSM-codec [5] gave improvements in SNR of 0.2 dB on average but 3 dB for sine signals. The iteration converges such fast, that three to four filter matrix sets $(\mathbf{H}^0_\nu, \mathbf{H}^1_\nu)$ were sufficient to compute $\mathbf{c}^{opt}_\nu$. So the proposed solution can be applied without introducing additional delay.

In the case of vector quantization when $\mathbf{c}_\nu$ is a codevector taken from a codebook, analytical solutions can not be expected. But equation (9) and (10) can be interpreted in an instructive way: If the expected value of the future optimum excitation vector $E\{\mathbf{c}^{opt}_{\nu+1}|\cdots\}$ were given, the optimum $\mathbf{c}^{opt}_\nu$ could be computed by minimizing the special squared error over two frames as noted in (9).

Using estimated values $\hat{\mathbf{c}}^{opt}_{\nu+1}$ for $E\{\mathbf{c}^{opt}_{\nu+1}|\cdots\}$ provides the basis to develop new improved sequential quantization methods as an approximation to the theoretic solution. One proposed scheme is the following: If one determines $\hat{\mathbf{c}}^{opt}_{\nu+1}$ with the assumption $\mathbf{r}_{\nu+2}-\mathbf{c}_{\nu+2} = \mathbf{0}$ from minimizing

$$\|\mathbf{H}^1_{\nu+1}(\mathbf{r}_\nu-\mathbf{c}_\nu) + \mathbf{H}^0_{\nu+1}(\mathbf{r}_{\nu+1}-\hat{\mathbf{c}}^{opt}_{\nu+1})\|^2 + \|\mathbf{H}^1_{\nu+2}(\mathbf{r}_{\nu+1}-\hat{\mathbf{c}}^{opt}_{\nu+1})\|^2 \tag{13}$$

and uses this $\hat{\mathbf{c}}^{opt}_{\nu+1}$ for determination of $\hat{\mathbf{c}}^{opt}_\nu$ from (9) and (10) an approximate solution for $\mathbf{c}^{opt}_\nu$ can be found.

This must be done by iteration since in (13) $\mathbf{c}_\nu$ is already included. It starts with $\mathbf{r}_\nu-\mathbf{c}_\nu = \mathbf{0}$ in (13) then (9) is used to find an iteration value of $\hat{\mathbf{c}}^{opt}_\nu$ which in turn can be used in (13) to improve the iteration value of $\hat{\mathbf{c}}^{opt}_{\nu+1}$. If $\hat{\mathbf{c}}^{opt}_\nu$ does not change any more (fixpoint), the iteration stops and $\hat{\mathbf{c}}^{opt}_\nu$ is taken as the quantized excitation sequence for frame I_ν.

Using the proposed algorithm will raise SNR without changing an existing decoder or the channel structure.

5. SUMMARY AND CONCLUSIONS

In this paper we have addressed the problem of analysis-by-synthesis quantization in which the quantization error depends on several excitation vectors together that must be determined sequentially, one after the other. This type of quantization is typical for state-of-the-art speech coders but may also occur in other fields. Especially, we were interested in the question to what extent information about the future original signal can improve quantization.

The task of coupled optimization was systematically described. It falls in the area of calculus of variations. After analyzing, a necessary condition for the optimum quantization rule was presented which is strong enough to determine explicit solution functions.

Thus a theoretical bound of the weighted mean square error has been derived for realistic conditions. This allows to judge how far a specific analysis-by-synthesis coding scheme approaches the theoretical optimum.

Application of the implicit criterion to a fixed grid RPE-approach led to an analytical solution. It was implemented in a simulation of the GSM-codec and gained some improvement as stated by theory. This approach supports e. g. compatible performance improvements of the GSM-codec for signaling tones and voice band data signals.

Exploitation of the theoretical results to other quantization methods is possible and was indicated for the vector quantization method in form of an approximation. This scheme is practical implemented too and currently under investigation.

APPENDIX

Starting from (8) and the minimum conditions we derive the necessary condition (10).

For $\delta_\nu(\cdots)$ the following recursion can be shown by separating the constant part not involved by the inner averaging and reducing the condition to effective elements.

$$\begin{aligned}\delta_\nu(\mathbf{r}_{\nu-1}-\mathbf{c}_{\nu-1},\mathbf{r}_\nu-\mathbf{c}_\nu,\mathbf{r}_{\nu+1},\mathbf{p}_{\nu+1},\ldots,\mathbf{p}_M) = \\ \|\mathbf{H}^1_\nu(\mathbf{r}_{\nu-1}-\mathbf{c}_{\nu-1}) + \mathbf{H}^0_\nu(\mathbf{r}_\nu-\mathbf{c}_\nu)\|^2 + \\ E\{\delta_{\nu+1}(\mathbf{r}_\nu-\mathbf{c}_\nu,\mathbf{r}_{\nu+1}-\mathbf{p}_{\nu+1}(\mathbf{r}_\nu-\mathbf{c}_\nu,\mathbf{r}_{\nu+1},\mathbf{r}_{\nu+2}), \\ \mathbf{r}_{\nu+2},\mathbf{p}_{\nu+2},\ldots,\mathbf{p}_M)|\mathbf{r}_\nu-\mathbf{c}_\nu,\mathbf{r}_{\nu+1}\}\end{aligned} \tag{14}$$

In this recursion the expected value just corresponds to averaging over $\mathbf{r}_{\nu+2}$. Inserting a set of solution functions in the expression $\delta_\nu(\cdots)$, it takes the minimum value $\delta_\nu^{min}(\cdots)$ in which the dependency on $\varphi_1, \ldots, \varphi_M$ will be omitted.

As usual $\mathbf{R}$ denotes the set of real numbers. For given $\mathbf{r}_\nu, \mathbf{c}_\nu, \mathbf{r}_{\nu+1} \in \mathbf{R}^L$, $\delta_{\nu+1}^{min}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1} - \mathbf{c}_{\nu+1}, \mathbf{r}_{\nu+2})$ is minimized by taking $\mathbf{c}_{\nu+1} = \mathbf{c}_{\nu+1}^{opt} = \varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2})$. Thus $\mathbf{c}_{\nu+1} = \varphi_{\nu+1}(\mathbf{r}_\nu - \hat{\mathbf{c}}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2})$ with $\hat{\mathbf{c}}_\nu \in \mathbf{R}^L$ will yield a value greater or equal and

$$\begin{aligned} &\delta_{\nu+1}^{min}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \hat{\mathbf{c}}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}), \mathbf{r}_{\nu+2}) \geq \\ &\delta_{\nu+1}^{min}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}), \mathbf{r}_{\nu+2}) \end{aligned} \tag{15}$$

holds for all $\mathbf{c}_\nu, \hat{\mathbf{c}}_\nu \in \mathbf{R}^L$.

Taking expected values (which is just averaging over $\mathbf{r}_{\nu+2}$) on both sides does not change the inequation, which leads to

$$\begin{aligned} A_1(\mathbf{c}_\nu, \hat{\mathbf{c}}_\nu) &= \left\|\mathbf{H}_\nu^1(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}) + \mathbf{H}_\nu^0(\mathbf{r}_\nu - \mathbf{c}_\nu)\right\|^2 + E\{\delta_{\nu+1}^{min}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \hat{\mathbf{c}}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}), \mathbf{r}_{\nu+2}) | \mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_\nu - \hat{\mathbf{c}}_\nu, \mathbf{r}_{\nu+1}\} \\ &\geq \\ A_2(\mathbf{c}_\nu) &= \left\|\mathbf{H}_\nu^1(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}) + \mathbf{H}_\nu^0(\mathbf{r}_\nu - \mathbf{c}_\nu)\right\|^2 + E\{\delta_{\nu+1}^{min}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}), \mathbf{r}_{\nu+2}) | \mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}\} \\ &= \delta_\nu^{min}(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}, \mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}) \end{aligned} \tag{16}$$

for all $\mathbf{c}_\nu, \hat{\mathbf{c}}_\nu \in \mathbf{R}^L$.

Further application of (14) gives

$$\begin{aligned} &A_1(\mathbf{c}_\nu, \hat{\mathbf{c}}_\nu) = \\ &\left\|\mathbf{H}_\nu^1(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}) + \mathbf{H}_\nu^0(\mathbf{r}_\nu - \mathbf{c}_\nu)\right\|^2 + \\ &E\Big\{\left\|\mathbf{H}_{\nu+1}^1(\mathbf{r}_\nu - \mathbf{c}_\nu) + \mathbf{H}_{\nu+1}^0(\mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \hat{\mathbf{c}}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}))\right\|^2 + \\ &\quad E\{\delta_{\nu+2}^{min}(\mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \hat{\mathbf{c}}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}), \mathbf{r}_{\nu+2} - \varphi_{\nu+2}(\mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \hat{\mathbf{c}}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}), \mathbf{r}_{\nu+2}, \mathbf{r}_{\nu+3}), \mathbf{r}_{\nu+3}) | \mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \hat{\mathbf{c}}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}), \mathbf{r}_{\nu+2}\} \\ &\qquad \Big| \mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_\nu - \hat{\mathbf{c}}_\nu, \mathbf{r}_{\nu+1}\Big\} \\ &\geq \\ &A_2(\mathbf{c}_\nu) = \\ &\left\|\mathbf{H}_\nu^1(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}) + \mathbf{H}_\nu^0(\mathbf{r}_\nu - \mathbf{c}_\nu)\right\|^2 + \\ &E\Big\{\left\|\mathbf{H}_{\nu+1}^1(\mathbf{r}_\nu - \mathbf{c}_\nu) + \mathbf{H}_{\nu+1}^0(\mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}))\right\|^2 + \\ &\quad E\{\delta_{\nu+2}^{min}(\mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}), \mathbf{r}_{\nu+2} - \varphi_{\nu+2}(\mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}), \mathbf{r}_{\nu+2}, \mathbf{r}_{\nu+3}), \mathbf{r}_{\nu+3}) | \mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}), \mathbf{r}_{\nu+2}\} \\ &\qquad \Big| \mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1}\Big\}. \end{aligned} \tag{17}$$

As presupposed $\delta_\nu^{min}(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}, \mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_{\nu+1})$ takes its minimal value for $\mathbf{c}_\nu = \mathbf{c}_\nu^{opt} = \varphi_\nu(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}, \mathbf{r}_\nu, \mathbf{r}_{\nu+1})$. Thus

$$A_1(\mathbf{c}_\nu, \hat{\mathbf{c}}_\nu) \geq A_2(\mathbf{c}_\nu) \geq A_2(\mathbf{c}_\nu^{opt}) \quad \Longrightarrow \quad A_1(\mathbf{c}_\nu, \hat{\mathbf{c}}_\nu) \geq A_2(\mathbf{c}_\nu^{opt}) \quad \text{for all } \mathbf{c}_\nu, \hat{\mathbf{c}}_\nu \in \mathbf{R}^L. \tag{18}$$

If now $\hat{\mathbf{c}}_\nu = \mathbf{c}_\nu^{opt}$ is chosen, the last terms on either side in (17) become equal and

$$\begin{aligned} &\left\|\mathbf{H}_\nu^1(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}) + \mathbf{H}_\nu^0(\mathbf{r}_\nu - \mathbf{c}_\nu)\right\|^2 + E\Big\{\left\|\mathbf{H}_{\nu+1}^1(\mathbf{r}_\nu - \mathbf{c}_\nu) + \mathbf{H}_{\nu+1}^0(\mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu^{opt}, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}))\right\|^2 \Big| \mathbf{r}_\nu - \mathbf{c}_\nu, \mathbf{r}_\nu - \mathbf{c}_\nu^{opt}, \mathbf{r}_{\nu+1}\Big\} \geq \\ &\left\|\mathbf{H}_\nu^1(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}) + \mathbf{H}_\nu^0(\mathbf{r}_\nu - \mathbf{c}_\nu^{opt})\right\|^2 + E\Big\{\left\|\mathbf{H}_{\nu+1}^1(\mathbf{r}_\nu - \mathbf{c}_\nu^{opt}) + \mathbf{H}_{\nu+1}^0(\mathbf{r}_{\nu+1} - \varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu^{opt}, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}))\right\|^2 \Big| \mathbf{r}_\nu - \mathbf{c}_\nu^{opt}, \mathbf{r}_{\nu+1}\Big\} \end{aligned} \tag{19}$$

follows for all $\mathbf{c}_\nu \in \mathbf{R}^L$.

The following equality $E\left\{\|\mathbf{K} - \mathbf{x}\|^2\right\} = \|\mathbf{K} - E\{\mathbf{x}\}\|^2 + \mathrm{Var}\{\mathbf{x}\}$ for the random value $\mathbf{x}$ and the constant $\mathbf{K}$ can be used in (19) to subtract the term that corresponds to $\mathrm{Var}\{\mathbf{x}\}$ on both sides, which results in

$$\begin{aligned} A(\mathbf{c}_\nu, \mathbf{c}_\nu^{opt}) = &\left\|\mathbf{H}_\nu^1(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}) + \mathbf{H}_\nu^0(\mathbf{r}_\nu - \mathbf{c}_\nu)\right\|^2 + \left\|\mathbf{H}_{\nu+1}^1(\mathbf{r}_\nu - \mathbf{c}_\nu) + \mathbf{H}_{\nu+1}^0(\mathbf{r}_{\nu+1} - E\{\varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu^{opt}, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}) | \mathbf{r}_\nu - \mathbf{c}_\nu^{opt}, \mathbf{r}_{\nu+1}\})\right\|^2 \geq \\ &\left\|\mathbf{H}_\nu^1(\mathbf{r}_{\nu-1} - \mathbf{c}_{\nu-1}) + \mathbf{H}_\nu^0(\mathbf{r}_\nu - \mathbf{c}_\nu^{opt})\right\|^2 + \left\|\mathbf{H}_{\nu+1}^1(\mathbf{r}_\nu - \mathbf{c}_\nu^{opt}) + \mathbf{H}_{\nu+1}^0(\mathbf{r}_{\nu+1} - E\{\varphi_{\nu+1}(\mathbf{r}_\nu - \mathbf{c}_\nu^{opt}, \mathbf{r}_{\nu+1}, \mathbf{r}_{\nu+2}) | \mathbf{r}_\nu - \mathbf{c}_\nu^{opt}, \mathbf{r}_{\nu+1}\})\right\|^2 \end{aligned} \tag{20}$$

for all $\mathbf{c}_\nu \in \mathbf{R}^L$ and (10) follows.

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