LEAST-SQUARES DESIGN OF SUBSAMPLED ALLPASS TRANSFORMED DFT FILTER-BANKS WITH LTI PROPERTY

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ABSTRACT

A new design approach for an allpass transformed analysis-synthesis filter-bank (AS FB) with subsampling is proposed, which can be used for adaptive subband processing with non-uniform time-frequency resolution. Signal reconstruction is performed by an FIR synthesis filter-bank whose coefficients are determined by two different linear least-squares (LS) error designs. The presented unconstrained LS optimization leads to perfect reconstruction (PR), but an insufficient bandpass characteristic for the synthesis subband filters. This problem is solved by a second, equality constrained least-squares (CLS) error design, which achieves near-perfect reconstruction (NPR). In the process, the linear distortions are minimized with the constraint for a linear, time-invariant (LTI) overall transfer function. The aliasing-free reconstruction error is significantly lower than for known designs of allpass transformed DFT filter-banks with NPR. Moreover, the new filter-bank design allows for a low and adjustable signal delay.

Index Terms— allpass transformation, frequency warping, DFT filter-bank, LTI system, least-squares design

1. INTRODUCTION

The allpass transformation is a well-known technique to realize a digital filter-bank with non-uniform time-frequency resolution [1, 2, 3]. Such a frequency warped filter-bank can mimic the Bark frequency scale, which models the frequency resolution of the human ear, with great accuracy [4]. This property is exploited, among others, for speech and audio processing algorithms, e.g., [5].

The original design applies the allpass transformation to an FIR analysis-synthesis filter-bank (AS FB) which leads to an IIR filter-bank [3]. The aliasing distortions can be reduced, but not canceled, by an improved prototype filter design, e.g., [6]. The phase distortions due to the allpass transformation can be (partly) compensated by a phase equalizer at the filter-bank output [5, 7].

An alternative is to use FIR synthesis filters instead of IIR filters. There exists an analytical closed-form solution for an FIR synthesis filter-bank to achieve perfect reconstruction (PR) for an allpass transformation of first order. This is shown in [8] for critical subsampling and in [9, 10] for arbitrary subsampling rates. A severe drawback of this solution is the missing bandpass characteristic of the synthesis filters [10]. This causes high signal reconstruction errors, if subband processing, such as spectral weighting or quantization, takes place.

A significantly better frequency selectivity exhibits the FIR synthesis filter-bank proposed in [11], which achieves near-perfect reconstruction (NPR). The phase distortions, introduced by the allpass transformation of the analysis filter-bank, are compensated by FIR phase equalizers designed via simple closed-form expressions. The aliasing distortions are reduced by prototype filters of high order. The lifting scheme can be employed to constrain the overall signal delay [11]. However, the analysis and synthesis filter-bank of this design are operated at the non-decimated sampling rate which causes a very high computational complexity. Moreover, aliasing distortions remain and the filter-bank has still a comparatively high signal delay due to the employed FIR phase equalizers.

In this contribution, a new design approach for an allpass transformed DFT AS FB is proposed, which ensures a linear, time-invariant (LTI) overall transfer function despite subsampling. In a first version, the FIR synthesis filters are determined by an unconstrained least-squares (LS) error design, which yields PR but an insufficient bandpass characteristic for the synthesis filter. This problem is addressed by a second, constrained least-squares (CLS) error design. The obtained filter-bank with NPR is still an LTI system, which has a low signal delay and causes only a small amount of linear distortions.

2. THE FREQUENCY WARPED DFT FILTER-BANK

The allpass transformed DFT filter-bank [1, 2, 3] is a generalization of the uniform DFT filter-bank. This frequency warped filter-bank has a non-uniform time-frequency resolution and is obtained by substituting the delay elements of the (uniform) FIR subband filters by allpass filters: $z^{-1} \rightarrow H_A(z)$. For this *allpass transformation*, an allpass filter of first order is considered with transfer function

$$H_A(z) = \frac{1-\alpha z}{z-\alpha}; \quad |z| > |\alpha|; \quad |\alpha| < 1; \quad \alpha \in \mathbb{R} \, . \qquad (1)$$

The allpass transformed analysis subband filters are given by

$$\widetilde{H}_i(z) = \sum_{n=0}^{M-1} h(n) \, W_M^{n\,i} \, H_A(z)^n ; \quad i \in \{0, 1, \dots, M-1\} \, . \quad (2)$$

The analysis prototype filter has the finite impulse response (FIR) $h(n)$ of length M. The complex modulation factor is denoted by

$$W_M = \exp\{-j\,2\pi/M\} \ . \qquad (3)$$

The efficient implementation of this analysis filter-bank with downsampling is shown in Fig. 1. The discrete Fourier transform (DFT) can be calculated efficiently by the fast Fourier transform (FFT). The FIR synthesis subband filters are given by

$$\bar{G}_i(z) = \sum_{\rho=0}^{M-1} g(M-1-\rho) \, W_M^{-i\,\rho} \underbrace{\sum_{\lambda=0}^{N_{\mathrm{p}}-1} p(\lambda,\rho) \, z^{-\lambda}}_{= P(z,\rho)} \qquad (4)$$

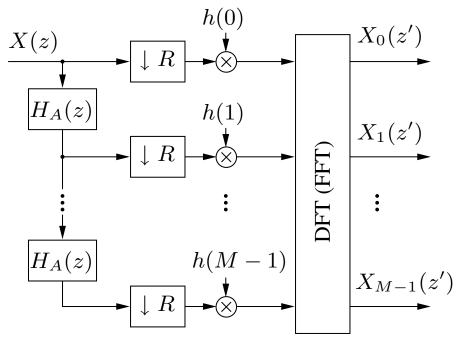

Fig. 1. Allpass transformed DFT analysis filter-bank with down-sampling.

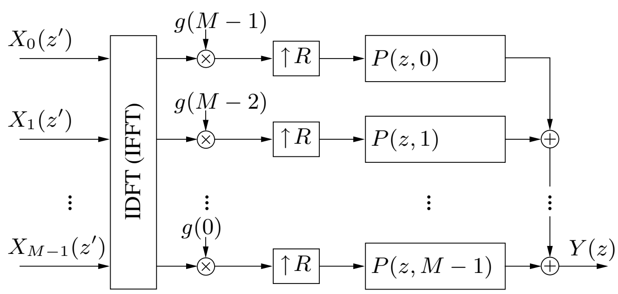

Fig. 2. DFT synthesis filter-bank with upsampling.

with $g(n)$ denoting the impulse response of the synthesis prototype filter. The M filters with transfer functions $P(z,\rho)$ and filter degree $N_\mathrm{p}-1$ shall be designed to achieve (almost) perfect reconstruction. The described synthesis filter-bank with upsampling by R is illustrated in Fig. 2.

For this warped DFT AS FB, the input-output relation in the z-domain reads

$$Y(z) = \frac{1}{R}\sum_{r=0}^{R-1} X(z\,W_R^r)\sum_{i=0}^{M-1}\widetilde{H}_i(z\,W_R^r)\,\bar{G}_i(z)\,. \tag{5}$$

In general, the AS FB (with subsampling) is a *linear, periodically time-varying* (LPTV) system with period R. To account for this behavior, we determine the overall transfer function of the filter-bank by R time-shifted unit samples sequences as input, i.e., $X(z)=z^{-l}$ for $l\in\{0,1,\ldots,R-1\}$. By this means, insertion of Eq. (2) and Eq. (4) into Eq. (5) yields the new transfer function

$$\begin{aligned} T_l(z) = \frac{Y(z)}{z^{-l}} &= \frac{1}{R}\sum_{r=0}^{R-1} W_R^{-r\,l}\sum_{i=0}^{M-1}\sum_{n=0}^{M-1} h(n)\,W_M^{n\,i}\,H_A(z\,W_R^r)^n \\ &\cdot \sum_{\rho=0}^{M-1} g(M-1-\rho)\,W_M^{-i\,\rho}\sum_{\lambda=0}^{N_\mathrm{p}-1} p(\lambda,\rho)\,z^{-\lambda}\,. \end{aligned} \tag{6}$$

The *uniform* DFT filter-bank is included as special case for $\alpha=0$ and $P(z,\rho)=z^{-(M-1-\rho)}$.

For the new filter-bank design, a matrix representation of the transfer function $T_l(z)$ in dependence of the unknown $N_\mathrm{p}\,M$ filter coefficients $p(\lambda,\rho)$ is required. In the following, bold lower-case variables mark vectors and matrices are denoted by bold upper-case variables. The superscripts T and H mark the transpose and conjugate transpose of a matrix, respectively. Eq. (6) can be expressed by the matrix notation

$$T_l(z) = \frac{1}{R}\sum_{r=0}^{R-1} W_R^{-r\,l}\sum_{i=0}^{M-1}\widetilde{H}_i(z\,W_R^r)\cdot\underbrace{\boldsymbol{v}_i^T\cdot\boldsymbol{D}(z)^T\cdot\boldsymbol{p}}_{=\bar{G}_i(z)} \tag{7}$$

$$\text{with}\ \ \boldsymbol{v}_i = [\,g(M-1),\ g(M-2)\,W_M^{-i},\ldots,g(0)\,W_M^{-i\,(M-1)}\,]^T \tag{8}$$

$$\boldsymbol{D}(z) = \boldsymbol{I}_M\otimes\boldsymbol{d}_{N_\mathrm{p}}(z) \tag{9}$$

$$\boldsymbol{d}_{N_\mathrm{p}}(z) = [\,1,\ z^{-1},\ \ldots,\ z^{-(N_\mathrm{p}-1)}\,]^T \tag{10}$$

$$\boldsymbol{p} = [\boldsymbol{p}_0^T,\ \boldsymbol{p}_1^T,\ \ldots,\ \boldsymbol{p}_{M-1}^T\,]^T \tag{11}$$

$$\boldsymbol{p}_\rho = [\,p(0,\rho),\ p(1,\rho),\ \ldots,\ p(N_\mathrm{p}-1,\rho)\,]^T \qquad i,\rho\in\{0,1,\ldots,M-1\}\,. \tag{12}$$

The $M\times M$ identity matrix is denoted by $\boldsymbol{I}_M$ and $\otimes$ marks the Kronecker product of two matrices. The transfer function of Eq. (7) can now be formulated by the compact notation

$$T_l(z) \doteq \boldsymbol{\xi}(z,l)\cdot\boldsymbol{p}\,. \tag{13}$$

The introduced complex vector $\boldsymbol{\xi}(z,l)$ is of dimension $1\times M\,N_\mathrm{p}$.

3. UNCONSTRAINED LEAST-SQUARES DESIGN

A *linear, time-invariant* (LTI) AS FB with PR is obtained, if Eq. (6) fulfills the requirement

$$T_l(z) \stackrel{!}{=} z^{-d_0};\quad l\in\{0,1,\ldots,R-1\} \tag{14}$$

with d_0 marking the signal delay of the filter-bank. This condition can be rewritten by using the matrix notation of Eq. (13)

$$\underbrace{\begin{bmatrix}\boldsymbol{\xi}(z,0)\\ \boldsymbol{\xi}(z,1)\\ \vdots\\ \boldsymbol{\xi}(z,R-1)\end{bmatrix}}_{=\boldsymbol{\Xi}_R(z)}\cdot\boldsymbol{p} \stackrel{!}{=} z^{-d_0}\cdot\mathbf{1}_R\,. \tag{15}$$

A column vector with R ones is denoted by $\mathbf{1}_R$. The $M\,N_\mathrm{p}$ unknown coefficients of the vector $\boldsymbol{p}$ are determined by the requirement that Eq. (15) should be fulfilled for the $K=M\,N_\mathrm{p}$ frequency points

$$z = W_K^n;\quad n\in\{0,1,\ldots,K-1\}\,. \tag{16}$$

With the short notation

$$\boldsymbol{A} = \begin{bmatrix}\boldsymbol{\Xi}_R(1)\\ \boldsymbol{\Xi}_R(W_K)\\ \vdots\\ \boldsymbol{\Xi}_R(W_K^{K-1})\end{bmatrix}\in\mathbb{C}^{K\,R\times M\,N_\mathrm{p}} \tag{17}$$

$$\boldsymbol{b}(d_0) = \begin{bmatrix}\mathbf{1}_R\\ W_K^{-d_0}\cdot\mathbf{1}_R\\ \vdots\\ W_K^{-(K-1)\,d_0}\cdot\mathbf{1}_R\end{bmatrix}\in\mathbb{C}^{K\,R\times 1}, \tag{18}$$

Eq. (15) turns into an R-times overdetermined set of $K\,R$ linear equations

$$\boldsymbol{A}\cdot\boldsymbol{p} \stackrel{!}{=} \boldsymbol{b}(d_0)\,. \tag{19}$$

A *least-squares* (LS) error solution is given by (cf. [12])

$$\boldsymbol{p}_{\text{LS}} = \arg\min_{\boldsymbol{p}} \left|\left| \boldsymbol{A} \cdot \boldsymbol{p} - \boldsymbol{b}(d_0) \right|\right|_2 \tag{20}$$

$$= \boldsymbol{A}^{\#} \cdot \boldsymbol{b}(d_0) \tag{21}$$

with $\boldsymbol{A}^{\#}$ denoting the pseudo-inverse matrix.[1] This LS design provides a warped AS FB with PR for an appropriate parameter configuration, for example, $N_{\text{p}} = 2R - 1$ and $d_0 = R - 1$. However, it turns out that the obtained synthesis subband filters have often no bandpass characteristic. This problem can also be observed for the closed-form PR designs of [9, 8, 10]. An advantage of the new LS design is that signal delay d_0 and filter length N_{p} can be controlled. Moreover, it allows the following modification to obtain synthesis filter with pronounced bandpass characteristic.

4. CONSTRAINED LEAST-SQUARES DESIGN

In order to incorporate constraints on the synthesis filter-bank design, we relax the requirement of Eq. (14) by the less strict demand for an LTI filter-bank. This condition can be expressed by the introduced transfer function of Eq. (6) according to

$$T_l(z) \stackrel{!}{=} T_0(z)\,; \quad l \in \{1, 2, \ldots, R-1\}\,. \tag{22}$$

This requirement can be cast into a matrix representation by means of Eq. (13)

$$\underbrace{\begin{bmatrix} \boldsymbol{\xi}(z,1) - \boldsymbol{\xi}(z,0) \\ \boldsymbol{\xi}(z,2) - \boldsymbol{\xi}(z,0) \\ \vdots \\ \boldsymbol{\xi}(z,R-1) - \boldsymbol{\xi}(z,0) \end{bmatrix}}_{= \Xi_{\Delta}(z)} \cdot \boldsymbol{p} \stackrel{!}{=} \mathbf{0}_{R-1}, \tag{23}$$

where a column vector with R zeros is denoted by $\mathbf{0}_R$. This condition for an LTI system ensures an *aliasing-free* signal reconstruction. In this case, the transfer function of Eq. (6) becomes equal to the linear (time-invariant) transfer function of the filter-bank given by

$$T_{\text{lin}}(z) = \frac{1}{R} \sum_{i=0}^{M-1} \sum_{n=0}^{M-1} \sum_{\rho=0}^{M-1} W_M^{i\,(n-\rho)} h(n)\, g(M-1-\rho) \cdot H_A(z)^n\, P(z,\rho)$$

$$= \frac{M}{R} \sum_{n=0}^{M-1} h(n)\, g(M-1-n)\, H_A(z)^n\, P(z,n)\,. \tag{24}$$

Perfect reconstruction is obtained, if

$$\frac{M}{R} \sum_{n=0}^{M-1} h(n)\, g(M-1-n) \stackrel{!}{=} 1 \quad \text{and} \tag{25}$$

$$H_A(z)^n\, P(z,n) \stackrel{!}{=} z^{-d_0}\,; \quad n \in \{0, 1, \ldots, M-1\}\,. \tag{26}$$

The first requirement is easily fulfilled. The second one can be expressed by means of the matrices introduced in Section 2

$$\begin{bmatrix} 1 \\ H_A(z) \\ \vdots \\ H_A(z)^{M-1} \end{bmatrix} \odot \left(\boldsymbol{D}(z)^T \cdot \boldsymbol{p} \right) \doteq \boldsymbol{U}(z) \cdot \boldsymbol{p} \stackrel{!}{=} \underbrace{z^{-d_0} \cdot \mathbf{1}_M}_{= \boldsymbol{v}(z, d_0)} \tag{27}$$

[1] A solution exists but is non-unique if $\boldsymbol{A}^H \boldsymbol{A}$ is a singular matrix.

with $\odot$ denoting the element-wise multiplication of two matrices with the same dimension.

The conditions of Eq. (23) and Eq. (27) should be fulfilled simultaneously for the K discrete frequency points of Eq. (16). Therefore, we introduce the (stacking) notation

$$\boldsymbol{U}^{[K]} = \begin{bmatrix} \boldsymbol{U}(1) \\ \boldsymbol{U}(W_K) \\ \vdots \\ \boldsymbol{U}(W_K^{K-1}) \end{bmatrix}; \quad \boldsymbol{v}^{[K]}(d_0) = \begin{bmatrix} \mathbf{1}_M \\ W_K^{-d_0} \cdot \mathbf{1}_M \\ \vdots \\ W_K^{-d_0\,(K-1)} \cdot \mathbf{1}_M \end{bmatrix}. \tag{28}$$

In the same manner, the matrix $\Xi_{\Delta}^{[K]}$ is derived from the matrix $\Xi_{\Delta}(z)$ of Eq. (23). The K filter coefficients $\boldsymbol{p}$ to fulfill Eq. (23) and Eq. (27) can now be determined by the *equality constrained least-squares (CLS)* error optimization

$$\boldsymbol{p}_{\text{CLS}} = \arg\min_{\boldsymbol{p}} \left|\left| \boldsymbol{U}^{[K]} \cdot \boldsymbol{p} - \boldsymbol{v}^{[K]}(d_0) \right|\right|_2$$

$$\text{subject to} \quad \Xi_{\Delta}^{[K]} \cdot \boldsymbol{p} = \mathbf{0}_{(R-1)\,K}\,. \tag{29}$$

This optimization can be performed, e.g., by using the function `lsqlin` of the MATLAB optimization toolbox. With this design, linear distortions are minimized under the constraint of complete aliasing cancellation. This constrained optimization provides, in contrast to the unconstrained LS solution of Section 3, synthesis subband filters with pronounced bandpass characteristic as exemplified in the following.

5. DESIGN EXAMPLE

The design of a warped DFT AS FB with $M = 16$ channels, rectangular prototype filters, and an allpass coefficient of $\alpha = 0.4$ is considered for a subsampling rate of $R = M/4 = 4$. A filter length of $N_{\text{p}} = 36$ and a signal delay of $d_0 = 36$ are used for the CLS optimization.

A filter-bank is a multi-rate system which can be analyzed by its system response function $t_{\text{bi}}(k_1, k_2)$, which is the response of the system at time instant k_1 to a unit sample sequence at k_2. The corresponding two-dimensional frequency-domain representation is given by the bifrequency system function $T_{\text{bi}}(e^{j\,\Omega_1}, e^{j\,\Omega_2})$ [13], which is plotted in Fig. 3 for the new CLS filter-bank design. It shows an LTI system since no side-diagonals with aliasing components occur. The main diagonal corresponds to the linear transfer function of the filter-bank. The low amount of linear distortions is shown by the second subplot in greater detail. The third subplot shows that the synthesis filters exhibit a distinctive bandpass characteristic. (An increase of the bandwidth due to the frequency warping can also be observed.) The proposed unconstrained LS design leads to a PR filter-bank for the considered parameter setting, but does not achieve such a bandpass characteristic for the synthesis filters.

For a comparison, the FIR synthesis filter-bank design of [11] is considered. It yields the bifrequency system response of Fig. 4 for an overall system delay of $d_0 = 64$ samples. The linear distortions on the main diagonal are about ± 1.7 dB. These distortions become significantly higher for a delay of $d_0 = 36$ as used for the CLS design. Distinct aliasing components can also be observed on the side-diagonals, which are not eliminated by synthesis subband filters with an even higher filter degree, cf. [11]. Thus, this filter-bank design causes a higher signal reconstruction error despite a higher system delay and a higher algorithmic complexity in comparison to the proposed CLS design.

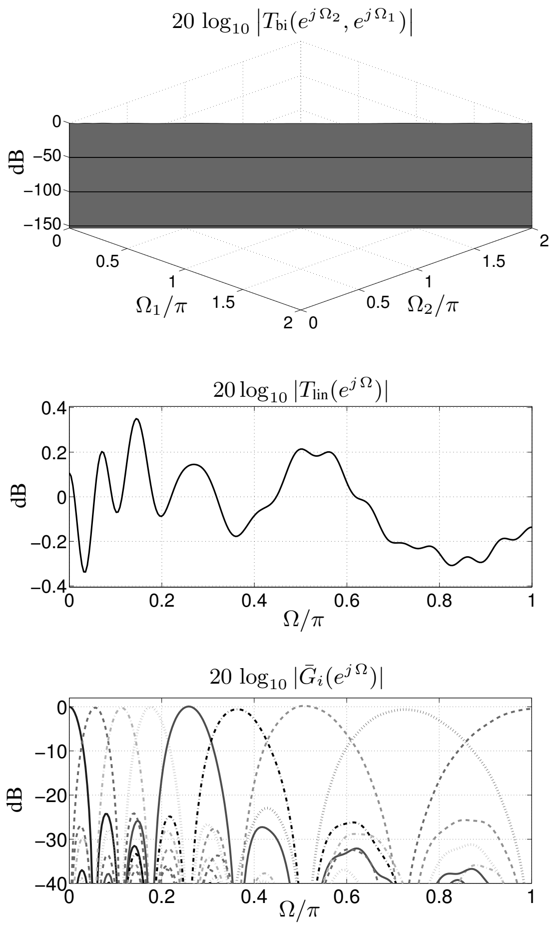

Fig. 3. Magnitude of the bifrequency system function $T_{\mathrm{bi}}(e^{j\,\Omega_1}, e^{j\,\Omega_2})$, the linear transfer function $T_{\mathrm{lin}}(e^{j\,\Omega})$ and the synthesis subband filters $\bar{G}_i(e^{j\,\Omega})$ obtained by the new CLS filter-bank design with $R = M/4 = 4,\ \alpha = 0.4,\ d_0 = 36,\ N_{\mathrm{p}} = 36$.

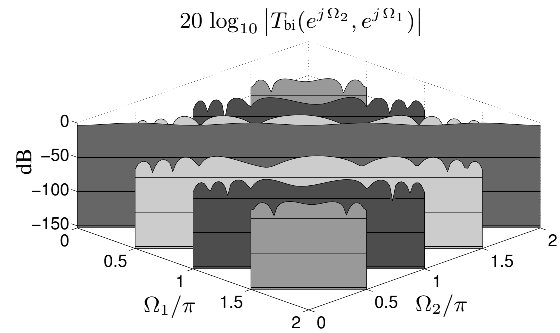

Fig. 4. Magnitude of the bifrequency system function for the warped DFT filter-bank of [11] with $R = M/4 = 4,\ \alpha = 0.4,\ d_0 = 64$.

6. CONCLUSIONS

A new design approach for frequency warped AS FBs is presented which ensures that the filter-bank is an LTI system despite subsampling. This is achieved by FIR synthesis filters whose coefficients are determined by two different linear least-squares error optimizations. The unconstrained optimization leads to PR, but an insufficient frequency selectivity for the synthesis subband filters. This problem, which can also be observed for existing closed-form PR design of warped AS FBs, is solved by an equality constrained least-squares error design. The obtained filter-bank is still an LTI system with adjustable signal delay, which causes only a small amount of linear distortions. The signal delay and aliasing-free reconstruction errors of the new filter-bank are significantly lower than for known designs of warped filter-banks with NPR. The proposed design allows to control the trade-off between signal delay and reconstruction errors, and it can also be applied to other transformation kernels than the considered DFT. The presented design for an allpass transformed DFT filter-bank is of special interest for speech and audio processing due to its non-uniform, if needed Bark scaled frequency bands.

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