

# Design of Multiple Order Allpass Transformed DFT Filter-Banks with Aliasing-Free Signal Reconstruction by Linear Programming

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## Abstract

The allpass transformation of multiple order is a very general approach to design a frequency warped analysis-synthesis filter-bank (AS FB) with non-uniform time-frequency resolution. In the design process, the delay elements of the analysis filter-bank are substituted by allpass filters of higher order to achieve a very flexible control over its frequency selectivity. Analytical closed-form designs for the synthesis filter-bank can ensure perfect reconstruction (PR), but the synthesis subband filters are not necessarily stable or exhibit a bandpass characteristic.

We tackle these problems by a constrained least-squares error (CLS) design. The coefficients of the new synthesis filter-bank are determined by a linear set of equations with linear constraints. The resulting synthesis filters are stable and exhibit a distinctive bandpass characteristic. Our design by linear programming can achieve an aliasing-free, near-perfect signal reconstruction, where the signal delay is an adjustable design parameter. These properties make the proposed filter-bank of interest for subband processing systems requiring non-uniform frequency bands.

## 1 Introduction

The allpass transformation is a well-known technique to design a digital filter-bank with non-uniform time-frequency resolution, e.g., [1–3]. Such frequency warped filter-banks are attractive for applications such as speech enhancement [4] or multiple description subband coding [5]. The most common design approach for these filter-banks is to use an allpass transformation of first order where the delay elements of the analysis filter-bank are substituted by allpass filters of first order. The synthesis filter-bank design for such frequency warped filter-banks is treated in several publications, e.g., [3, 4, 6–8].

A more general design approach is to employ an allpass transformation of multiple order where the delay elements of the analysis filter-bank are replaced by allpass filters of higher order [9, 10]. This concept provides a more flexible control over the time-frequency resolution, but complicates the synthesis filter-bank design. Several authors have derived an analytical closed-form design for the synthesis filter-bank to achieve perfect (signal) reconstruction (PR) [5, 9, 11]. However, the obtained IIR synthesis subband filters are in general either unstable or anti-causal, except for an allpass transformation of first order where FIR filters are obtained. Another severe drawback of this solution is the missing bandpass characteristic of the synthesis subband filters [11]. This causes high signal distortions if spectral modifications of the subband signals, such as spectral weighting or quantization, are performed.

In this contribution, we solve these problems by a novel design for a multiple order allpass transformed DFT analysis-synthesis filter-bank (AS FB). The presented constrained least-squares error (CLS) design yields FIR synthesis filters with pronounced bandpass characteristic. This design by linear programming achieves aliasing-free, near-perfect reconstruction with adjustable delay.

## 2 Multiple Order Allpass Transformed DFT Filter-Bank

Basis for our design is a uniform DFT filter-bank where the analysis subband filters are complex modulated versions of a prototype lowpass filter with finite impulse response (FIR)  $h(n)$  of length  $L$  according to

$$H_i(z) = H(z W_M^{-i}) = \sum_{n=0}^{L-1} h(n) W_M^{in} z^{-n} \quad (1)$$

with  $W_M = \exp\{-j2\pi/M\}$  and  $i \in \{0, 1, \dots, M-1\}$ , cf. [12]. A technique to achieve a non-uniform time-frequency resolution is to perform an *allpass transformation*, e.g., [1–3]. The most general approach for this is to replace the delay elements by allpass filters of *multiple order* [9, 10]. The allpass transformed subband filters are obtained by the substitution  $z^{-n} \rightarrow \Psi(z) \Theta^n(z)$ . This bilinear transformation converts Eq. (1) into the transfer function

$$\tilde{H}_i(z) = \Psi(z) \sum_{n=0}^{L-1} h(n) W_M^{in} \Theta^n(z) \quad (2)$$

$$\Psi(z) = \left( A_\beta^{[K-1]}(z) \right)^{L-1} \quad (3)$$

$$\Theta^n(z) = \left( \frac{A_\alpha^{[K]}(z)}{A_\beta^{[K-1]}(z)} \right)^n. \quad (4)$$

A causal, rational allpass filter of  $K$ -th order is given by

$$A_\alpha^{[K]}(z) = \prod_{k=1}^K \frac{1 - \alpha(k)z}{z - \alpha(k)} \quad (5a)$$

$$|\alpha(k)| < 1; \quad \alpha(k) \in \mathbb{R}; \quad \max_k \{|\alpha(k)|\} < z \quad (5b)$$

with frequency response  $A_\alpha^{[K]}(e^{j\Omega}) = \exp\{-j\varphi_\alpha^{[K]}(\Omega)\}$ . The frequency responses of the uniform subband filters of Eq. (1) and the non-uniform subband filters of Eq. (2) are related by

$$\tilde{H}_i(e^{j\Omega}) = e^{-j(L-1)\cdot\varphi_\beta^{[K-1]}(\Omega)} H_i\left(e^{j\tilde{\varphi}(\Omega)}\right) \quad (6)$$

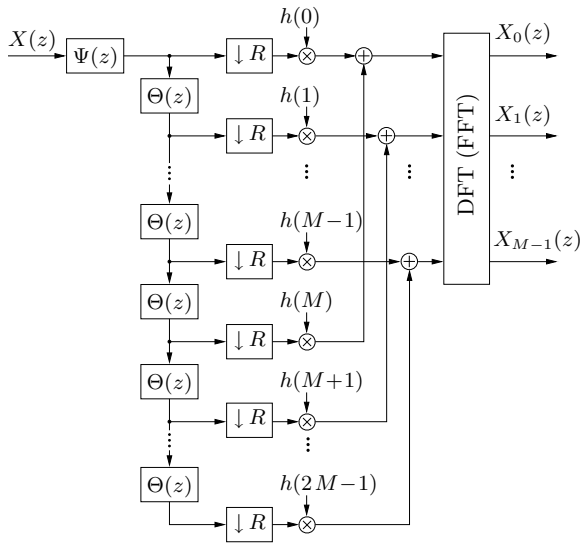
$$\text{with } \tilde{\varphi}(\Omega) = \varphi_\alpha^{[K]}(\Omega) - \varphi_\beta^{[K-1]}(\Omega). \quad (7)$$

The last equation reveals that the allpass transformation causes a *frequency warping* where a frequency interval of  $\Delta\Omega = 2\pi$  is mapped to an interval of  $2\pi$  on the warped frequency scale:

$$[0, 2\pi] \rightarrow [0, 2\pi] : \Omega \mapsto \tilde{\varphi}(\Omega). \quad (8)$$

This function is bijective, if the continuous (unwrapped) phase response  $\tilde{\varphi}(\Omega)$  is monotone increasing, which corresponds a positive group delay

$$\frac{\partial \tilde{\varphi}(\Omega)}{\partial \Omega} > 0. \quad (9)$$



**Figure 1:** Polyphase network (PPN) implementation of the multiple order allpass transformed DFT analysis filter-bank with downsampling by  $R$  for  $L = 2M$ .

Eq. (8) and (9) ensure subband filters without a comb filter structure [9,10]. A feasible choice is  $A_\beta^{[K-1]}(z) = z^{-(K-1)}$  so that only the allpass coefficients  $\alpha(k) \forall k \in \{1, \dots, K\}$  must be determined to achieve the desired frequency resolution and to fulfill Eq. (9), cf. [9].

An allpass transformation of *first order* is obtained for  $K = 1$  so that

$$\Psi(z) = 1 \quad \wedge \quad \Theta(z) = A_\alpha^{[1]}(z). \quad (10)$$

Eq. (2) can be rewritten

$$\tilde{H}_i(z) = \Psi(z) \sum_{m=0}^{l_M-1} \sum_{\lambda=0}^{M-1} h(mM + \lambda) \Theta^{mM+\lambda}(z) W_M^{\lambda i} \quad (11)$$

where it is assumed w.l.o.g. that  $L = l_M M$  with  $l_M$  being integer-valued. Fig. 1 shows the corresponding *polyphase network* (PPN) implementation of the analysis filter-bank, where the discrete Fourier transform (DFT) can be efficiently calculated by the fast Fourier transform (FFT).<sup>1</sup>

The FIR synthesis subband filters are here given by

$$\tilde{G}_i(z) = \sum_{n=0}^{L-1} g(n) W_M^{i(n+1)} P(z, L-1-n) \quad (12)$$

with  $g(n)$  denoting the FIR of the synthesis prototype filter of length  $L$ . The coefficients of the  $L$  transfer functions

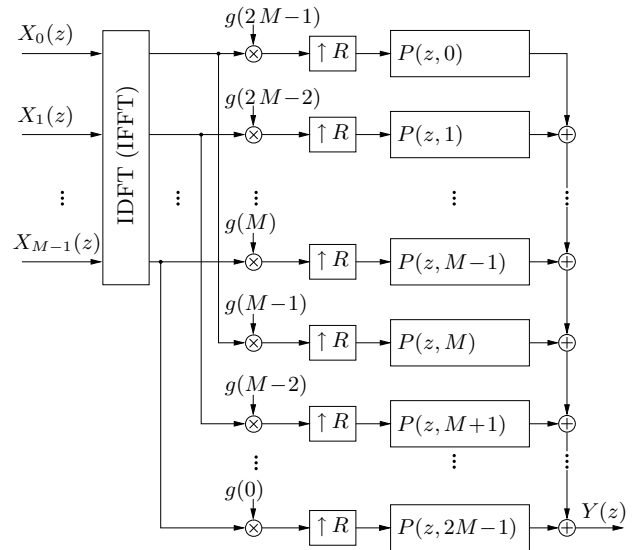
$$P(z, \lambda) = \sum_{n=0}^{N_p-1} p(n, \lambda) z^{-n}; \quad \lambda \in \{0, 1, \dots, L-1\} \quad (13)$$

shall be determined in such a way that a filter-bank with (almost) perfect reconstruction is obtained.

The synthesis subband filters can be expressed by the PPN representation

$$\tilde{G}_i(z) = \sum_{\lambda=0}^{M-1} \tilde{G}_{M-1-\lambda}^{(M)}(z) W_M^{-\lambda i}; \quad i \in \{0, 1, \dots, M-1\} \quad (14)$$

<sup>1</sup> This PPN implementation is not the most efficient one in case of an allpass transformation of higher order ( $K > 1$ ), but considered here for the sake of clarity.



**Figure 2:** PPN implementation of the DFT synthesis filter-bank with upsampling by  $R$  for  $L = 2M$ .

with 'modified' (type 1) polyphase components

$$\tilde{G}_\lambda^{(M)}(z) = \sum_{m=0}^{l_M-1} g(mM + \lambda) \cdot P(z, (l_M - m)M - 1 - \lambda). \quad (15)$$

This efficient PPN implementation of this synthesis filter-bank is shown in Fig. 2. The *uniform* DFT AS FB is included as special case for  $\Psi(z) = 1$ ,  $\Theta(z) = z^{-1}$  and  $P(z, n) = z^{-(L-1-n)}$  for  $n \in \{0, 1, \dots, L-1\}$ .

### 3 Synthesis Filter-Bank Design

The output signal of the filter-bank can be represented in the  $z$ -domain (for an input signal of finite energy) by the expression

$$Y(z) = \frac{1}{R} \sum_{r=0}^{R-1} X(z W_R^r) \sum_{i=0}^{M-1} \tilde{H}_i(z W_R^r) \cdot \tilde{G}_i(z). \quad (16)$$

It can be seen that the AS FB with subsampling by  $R$  is a *linear, periodically time-varying* (LPTV) system with period  $R$ . We will take this behavior into account by determining the overall transfer function of the filter-bank for  $R$  time-shifted unit sample sequences as input, i.e.,  $X(z) = z^{-l}$  for  $l \in \{0, 1, \dots, R-1\}$ . By this means, Eq. (16) leads to the new transfer function

$$T_l(z) = \frac{Y(z)}{z^{-l}} = \frac{1}{R} \sum_{r=0}^{R-1} W_R^{-r l} \sum_{i=0}^{M-1} \tilde{H}_i(z W_R^r) \cdot \tilde{G}_i(z). \quad (17)$$

For our filter-bank design, a matrix representation of the transfer function  $T_l(z)$  in dependence of the unknown  $L N_p$  filter coefficients  $p(n, \lambda)$  is required. In the following, bold lower-case variables denote vectors and matrices and are marked by bold upper-case variables. The superscript  $T$  indicates the transpose of a vector or matrix. Some manipulations of Eq. (12) lead to the matrix representation

$$\bar{G}_i(z) = \mathbf{v}_i^T \cdot \mathbf{D}(z)^T \cdot \mathbf{p}; \quad i \in \{0, 1, \dots, M-1\} \quad (18a)$$

with  $\mathbf{v}_i = [g(L-1), g(L-2)W_M^{-i}, \dots$

$$\dots, g(1)W_M^{-i(L-2)}, g(0)W_M^{-i(L-1)}]^T \quad (18b)$$

$$\mathbf{D}(z) = \mathbf{I}_L \otimes \mathbf{d}_{N_p}(z) \quad (18c)$$

$$\mathbf{d}_{N_p}(z) = [1, z^{-1}, \dots, z^{-(N_p-1)}]^T \quad (18d)$$

$$\mathbf{p} = [\mathbf{p}_0^T, \mathbf{p}_1^T, \dots, \mathbf{p}_{L-1}^T]^T \quad (18e)$$

$$\mathbf{p}_n = [p(0, n), p(1, n), \dots, p(N_p-1, n)]^T \quad (18f)$$

$$n \in \{0, 1, \dots, L-1\}. \quad (18g)$$

The  $L \times L$  identity matrix is denoted by  $\mathbf{I}_L$  and  $\otimes$  marks the Kronecker product of two matrices. The transfer function of Eq. (18) can now be formulated by the compact matrix notation

$$T_l(z) = \frac{1}{R} \sum_{r=0}^{R-1} W_R^{-rl} \sum_{i=0}^{M-1} \tilde{H}_i(z W_R^r) \cdot \mathbf{v}_i^T \cdot \mathbf{D}(z)^T \cdot \mathbf{p} \quad (19)$$

$$= \boldsymbol{\xi}(z, l) \cdot \mathbf{p}; \quad l \in \{0, 1, \dots, R-1\} \quad (20)$$

where the introduced complex vector  $\boldsymbol{\xi}(z, l)$  is of dimension  $1 \times LN_p$ .

The new representation for the transfer function according to Eq. (17) allows to express the condition for an aliasing-free signal reconstruction by the simple requirement

$$T_l(z) \stackrel{!}{=} T_0(z) \quad \text{for } l \in \{1, 2, \dots, R-1\}. \quad (21)$$

This condition can now be cast into a matrix notation by means of Eq. (20)

$$\underbrace{\begin{bmatrix} \boldsymbol{\xi}(z, 1) - \boldsymbol{\xi}(z, 0) \\ \boldsymbol{\xi}(z, 2) - \boldsymbol{\xi}(z, 0) \\ \vdots \\ \boldsymbol{\xi}(z, R-1) - \boldsymbol{\xi}(z, 0) \end{bmatrix}}_{= \boldsymbol{\Xi}_\Delta(z)} \cdot \mathbf{p} \stackrel{!}{=} \mathbf{0}_{R-1}, \quad (22)$$

where a column vector with  $R$  zeros is denoted by  $\mathbf{0}_R$ . This condition for a linear time-invariant (LTI) AS FB ensures an *aliasing-free* signal reconstruction. In this case, Eq. (17) becomes equal to the linear transfer function of the filter-bank, which can be expressed by Eq. (11) and (12) according to

$$T_{\text{lin}}(z) = \frac{1}{R} \Psi(z) \sum_{i=0}^{M-1} \sum_{n=0}^{L-1} \sum_{\rho=0}^{L-1} W_M^{i(n+\rho+1)} h(n) g(\rho) \cdot \Theta^n(z) P(z, L-1-\rho) \quad (23)$$

$$= \frac{M}{R} \Psi(z) \sum_{m \in \mathbb{Z}} \sum_{n=0}^{L-1} h(n) g(mM-1-n) \cdot \Theta^n(z) P(z, L-mM+n). \quad (24)$$

The condition  $T_{\text{lin}}(z) \stackrel{!}{=} z^{-d_0}$  avoids linear signal distortions and is fulfilled if

$$\frac{M}{R} \sum_{n=0}^{L-1} h(n) g(mM-1-n) = \begin{cases} 1 & ; m = l_M \\ 0 & ; m \in \mathbb{Z} \setminus \{l_M\} \end{cases} \quad (25)$$

$$\wedge \Psi(z) \cdot \Theta^n(z) \cdot P(z, n) = z^{-d_0} \quad \forall n \in \{0, 1, \dots, L-1\}. \quad (26)$$

Eq. (25) states a standard problem in the design of (uniform) AS FBs and can be either solved by analytical closed-form expressions or numerical designs approaches to better influence the frequency response, e.g., [12, 13].

The second requirement of Eq. (26) can be expressed by means of the matrices introduced in Eq. (18)

$$\Psi(z) \cdot \begin{bmatrix} 1 \\ \Theta(z) \\ \vdots \\ \Theta^{L-1}(z) \end{bmatrix} \odot (\mathbf{D}(z)^T \cdot \mathbf{p}) = \mathbf{U}(z) \cdot \mathbf{p} \stackrel{!}{=} \underbrace{z^{-d_0} \cdot \mathbf{1}_L}_{= \mathbf{v}(z, d_0)} \quad (27)$$

with  $\odot$  denoting the element-wise multiplication of two matrices of the same dimensions.

The conditions of Eq. (22) and Eq. (27) should be fulfilled for  $\mathcal{N} = LN_p$  discrete  $z$ -values on the unit circle

$$z = W_N^n; \quad n \in \{0, 1, \dots, \mathcal{N}-1\}. \quad (28)$$

Evaluating the matrix  $\mathbf{U}(z)$  and vector  $\mathbf{v}(z, d_0)$  of Eq. (27) at these points can be expressed by the (stacking) notation

$$\mathbf{U}^{[\mathcal{N}]} = \begin{bmatrix} \mathbf{U}(1) \\ \mathbf{U}(W_N) \\ \vdots \\ \mathbf{U}(W_N^{\mathcal{N}-1}) \end{bmatrix}; \quad \mathbf{v}^{[\mathcal{N}]}(d_0) = \begin{bmatrix} \mathbf{1}_L \\ W_N^{-d_0} \cdot \mathbf{1}_L \\ \vdots \\ W_N^{-d_0(\mathcal{N}-1)} \cdot \mathbf{1}_L \end{bmatrix}. \quad (29)$$

In the same manner, the matrix  $\boldsymbol{\Xi}_\Delta^{[\mathcal{N}]}$  is derived from the matrix  $\boldsymbol{\Xi}_\Delta(z)$  of Eq. (22). The  $\mathcal{N}$  filter coefficients  $\mathbf{p}$  to fulfill Eq. (22) and Eq. (27) can now be determined by the *equality constrained least-squares error* (CLS) problem

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p}} \left\| \mathbf{U}^{[\mathcal{N}]} \cdot \mathbf{p} - \mathbf{v}^{[\mathcal{N}]}(d_0) \right\|_2$$

subject to  $\boldsymbol{\Xi}_\Delta^{[\mathcal{N}]} \cdot \mathbf{p} = \mathbf{0}_{(R-1)\mathcal{N}}. \quad (30)$

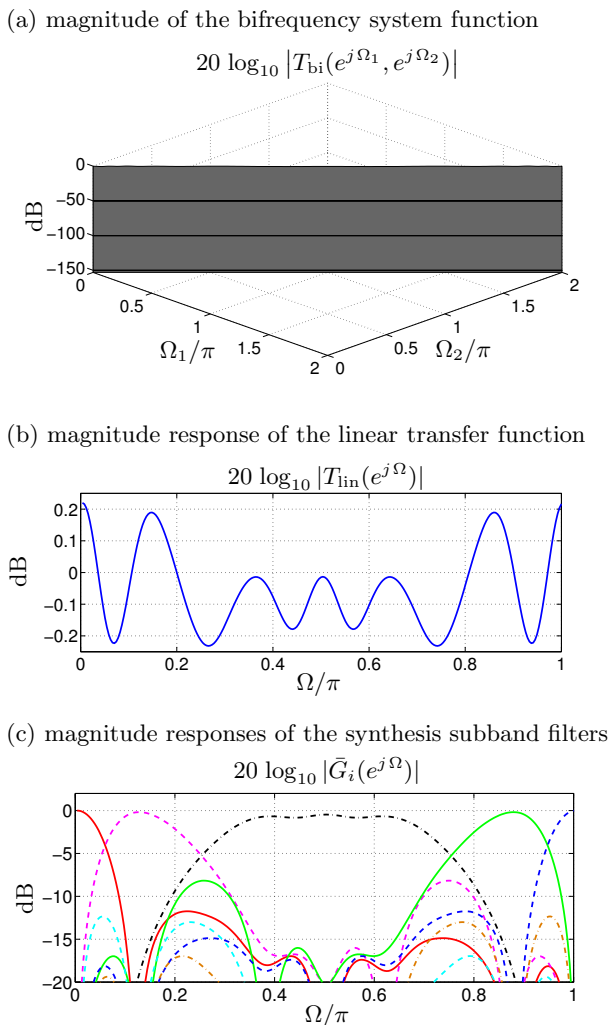
This linear programming can be solved by the function `lsqlin` of the MATLAB optimization toolbox. With this design, linear distortions are minimized under the constraint of complete aliasing cancellation. This constrained optimization provides synthesis subband filters with bandpass characteristic as exemplified later.

Conceptually, the proposed concept can be extended to an allpass transformation with complex allpass coefficients. However, this requires to solve a set of complex-valued equations and increases the implementation cost for the filter-bank considerably.

The proposed filter-bank design is very general and comprises some previous approaches as special cases. It can be shown that the design of [7] is obtained for an allpass transformation of first order according to Eq. (10) and if Eq. (30) is solved without constraints. The CLS design of [8] is included for an allpass transformation of first order and a prototype filter length of  $L = M$ . In this contribution, an unconstrained LS design to achieve PR is also devised. However, this design does not ensure synthesis subband filters with bandpass characteristic and is hence not considered here.

## 4 Design Example

The design of an AS FB with  $M = 8$  channels is considered with a subsampling rate of  $R = 4$  and rectangular prototype filter of length  $L = M$ . An allpass transformation is employed ( $K = 2$ ) with allpass coefficients  $\alpha(1) = -0.5$ ,  $\alpha(2) = 0.5$  and  $\beta(1) = 0$ . A filter length of  $N_p = 36$  and a delay of  $d_0 = 43$  are used for the CLS optimization of Eq. (30).



**Figure 3:** Analysis of new CLS filter-bank design with parameters  $L = M = 8$ ,  $R = 4$ ,  $\alpha(1) = -0.5$ ,  $\alpha(2) = 0.5$ ,  $\beta(1) = 0$ ,  $d_0 = 43$ ,  $N_p = 36$ .

A filter-bank is a multi-rate system whose transfer behavior can be analyzed by its bifrequency system function  $T_{\text{bi}}(e^{j\Omega_2}, e^{j\Omega_1})$ , cf. [13]. This function is plotted in Fig. 3-a for the new filter-bank design. It shows no side-diagonals which reveals a complete aliasing cancellation. The main-diagonal of the bifrequency system function represents the linear transfer function of the system. The negligible amount of linear distortions is shown in Fig. 3-b in greater detail. These linear distortions can be even further reduced by synthesis subband filters of higher degrees. The plot of the magnitude responses for the synthesis filters in Fig. 3-c shows their bandpass characteristic. The different bandwidths are caused by the frequency warping of the analysis filter-bank whose subband filters show a similar variation for their bandwidth.

## 5 Conclusions

A novel design for a non-uniform DFT AS FB with sub-sampling is presented whose time-frequency resolution is adjusted by an allpass transformation of multiple order. The coefficients of the FIR synthesis subband filters are determined by a linearly constrained least-squares error optimization. The obtained filter-bank can achieve a complete aliasing cancellation with a low amount of linear signal distortions. The FIR synthesis subband filter are stable and exhibit a distinctive bandpass characteristic, which

is not ensured by the closed-form PR filter-bank designs of [5,9,11]. Moreover, the signal delay is now an adjustable design parameter and not fix. The presented approach is very general and comprises the filter-bank designs [7,8] as special cases. In contrast to the designs of [5,9,11], our approach provides also a solution for a PPN filter-bank where the prototype filter length exceeds the number of channels ( $L > M$ ). The presented design by linear programming considers a DFT filter-bank, but it can be easily extended to other transformation kernels such as the discrete cosine transform (DCT). The flexible control of the frequency resolution for the devised filter-bank can be of interest for various applications including speech and audio subband processing systems.

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