Taking the derivative of (31) with respect to $H_{k,l}$, where $H_{k,l}$ is the entry of $\mathbf{H}$ located at row k and column l, based on the Wirtinger calculus [8], yields

$$\frac{\partial\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}}{\partial H_{kl}}\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)+\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{e}_k\mathbf{e}_l^T=\mathbf{0}. \quad (32)$$

Hence, after straightforward algebra, we have

$$\frac{\partial\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}}{\partial H_{kl}}=-\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\times\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{e}_k\mathbf{e}_l^T\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1} \quad (33)$$

yielding

$$\begin{aligned}\frac{\partial\mathbf{e}_i^T\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{e}_i}{\partial H_{kl}}&=-\mathbf{e}_i^T\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{e}_k\mathbf{e}_l^T\\&\quad\times\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{e}_i,\\&=-\mathbf{e}_l^T\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{e}_i\mathbf{e}_i^T\\&\quad\times\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{e}_k.\end{aligned} \quad (34)$$

Since by definition, $(\partial\mathbf{e}_i^T(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H})^{-1}\mathbf{e}_i)/\partial H_{kl}$ is the kth row and lth column entry of the matrix

$$-\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{e}_i\mathbf{e}_i^T\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^H\mathbf{R}_w^{-1}$$

then

$$\begin{aligned}\nabla_{\mathbf{H}}\mathbf{e}_i^H(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H})^{-1}\mathbf{e}_i&=-\mathbf{R}_w^{-1}\mathbf{H}\\&\quad\times(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H})^{-1}\mathbf{e}_i\mathbf{e}_i^T(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H})^{-1}.\end{aligned}$$

Using this in (30) yields

$$\begin{aligned}\nabla_{\mathbf{H}}J_{\mathrm{MMSE}}(\mathbf{H})&=-\sum_{i=1}^{M}\mathbf{R}_w^{-1}\mathbf{H}\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\\&\quad\times\mathbf{e}_i\mathbf{e}_i^T\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\\&=-\mathbf{R}_w^{-1}\mathbf{H}\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\mathbf{I}\\&\quad\times\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-1}\\&=-\mathbf{R}_w^{-1}\mathbf{H}\left(\mathbf{R}_x^{-1}+\mathbf{H}^H\mathbf{R}_w^{-1}\mathbf{H}\right)^{-2}.\end{aligned}$$

This completes the proof of Lemma 1. □

Lemma 2 [6, Ch. 2]: The inequality

$$\begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^H & \mathbf{R}\end{bmatrix}\geq 0 \quad (35)$$

where $\mathbf{Q}=\mathbf{Q}^H$, $\mathbf{R}=\mathbf{R}^H$, and $\mathbf{R}\geq 0$ is equivalent to

$$\mathbf{R}>0,\quad \mathbf{Q}-\mathbf{S}\mathbf{R}^{-1}\mathbf{S}^H\geq 0 \quad (36)$$

i.e., the set of nonlinear inequalities in (35) can be represented as (36).

Lemma 3 [3, Prop. 2]: Given matrices $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{A}$ with $\mathbf{A}=\mathbf{A}^H$

$$\mathbf{A}\geq\mathbf{P}^H\mathbf{Z}\mathbf{Q}+\mathbf{Q}^H\mathbf{Z}^H\mathbf{P},\quad\forall\|\mathbf{Z}\|\leq\alpha$$

if and only if there exists a $\lambda\geq 0$ such that

$$\begin{bmatrix}\mathbf{A}-\lambda\mathbf{Q}^H\mathbf{Q} & -\alpha\mathbf{P}^H\\ -\alpha\mathbf{P} & \lambda\mathbf{I}\end{bmatrix}\geq 0.$$

A proof of Lemma 3 is given in [3].

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Least-Squares Design of DFT Filter-Banks Based on Allpass Transformation of Higher Order

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***Abstract*—The allpass transformation of higher order is a very general concept to construct a frequency warped analysis–synthesis filter bank (AS FB) with nonuniform time-frequency resolution. In contrast to the more common allpass transformation of first order, the delay elements of the analysis filter bank are substituted by allpass filters of higher order to achieve a more flexible control over its frequency selectivity. Known analytical closed-form designs for the synthesis filter bank can ensure perfect reconstruction (PR), but the synthesis subband filters are not necessarily stable and exhibit no distinctive bandpass characteristic. These problems are addressed by a new least-squares error (LSE) filter bank design. The coefficients of the finite-impulse-response (FIR) synthesis filters are determined simply by a linear set of equations where the signal delay is an adjustable design parameter. This approach can achieve a perfect signal reconstruction with synthesis filters which are inherently stable and feature a bandpass characteristic. The proposed filter bank is of interest for various subband processing systems requiring nonuniform frequency bands.**

***Index Terms*—Allpass transformation, frequency warping, least squares, nonuniform filter banks, perfect reconstruction.**

I. Introduction

The allpass transformation is a common approach to design a filter bank with a nonuniform time-frequency resolution [1]–[3].

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Such frequency warped filter banks are beneficial for applications such as speech enhancement, e.g., [4], [5]. In addition, warped filter banks exhibit a lower signal delay and complexity than comparable nonuniform filter banks realized by a tree-structure, cf., [4].

The most common design approach for frequency warped filter banks is to use an allpass transformation of *first order* where the delay elements of the analysis filter bank are substituted by allpass filters of first order. The synthesis filter bank design for such allpass transformed filter banks is treated in several publications [4]–[10] which achieve a *nearly* perfect signal reconstruction.

A more general design approach is to employ an allpass transformation of *higher order* where the delay elements of the analysis filter bank are replaced by allpass filters of higher order [11], [12]. This concept provides a more flexible control over the time-frequency resolution, which is exploited, e.g., for multiple description subband coding [13], but it also complicates the synthesis filter bank design. An analytical closed-form design for the synthesis filter bank to achieve *perfect reconstruction* (PR) has been derived by several authors. In [12], a tree-structured network is proposed for signal reconstruction. However, a causal and stable synthesis filter bank has only been found for the special case of an allpass transformation of first order. In [13] and [14], the synthesis subband filters are derived by inversion of the aliasing component matrix. This yields infinite-impulse-response (IIR) synthesis subband filters which are not necessarily stable and causal, except for the special case of an allpass transformation of first order where finite-impulse–response (FIR) filters are obtained. Another severe drawback of all these closed-form solutions is that they do not ensure a bandpass characteristic for the synthesis subband filters, even if only an allpass transformation of first order is applied, cf., [14]. This can cause high signal distortions if spectral modifications of the subband signals are performed.

In this contribution, we tackle these problems by a novel design concept for a frequency warped DFT analysis–synthesis filter bank (AS FB) based on an allpass transformation of first or higher order. The presented least-squares error (LSE) design yields FIR synthesis filters with a distinctive bandpass characteristic and can achieve (almost) PR with adjustable signal delay.

This correspondence is organized as follows. In Section II, the considered nonuniform DFT filter bank based on an allpass transformation of higher order is introduced. The new synthesis filter bank design is presented in Section III. A comparison with related PR designs is provided in Section IV, and the main results of this contribution are summarized in Section V.

II. Allpass Transformed DFT Filter Bank

Basis for our design is a uniform DFT filter bank where the analysis subband filters are complex modulated versions of a prototype lowpass filter with FIR $h(n)$ of length L according to

$$H_i(z) = H\left(z\,W_M^i\right) = \sum_{n=0}^{L-1} h(n)\,W_M^{-in} z^{-n} \qquad \forall i \in \{0,1,\ldots,M-1\} \tag{1}$$

with $W_M = \exp\{-j2\pi/M\}$, cf., [15]. A well-known technique to achieve a nonuniform time-frequency resolution is to perform an *allpass transformation* where the delay elements of the subband filters are replaced by allpass filters of *first order* [1]–[3].

A more general approach is to substitute the delay elements by allpass filters of *higher order* [11], [12]:

$$z^{-n} \rightarrow A^n(z) \cdot B^{L-1-n}(z) \tag{2}$$

where $A(z)$ and $B(z)$ represent the transfer functions of stable causal allpass filters of order K and $K-1$ according to

$$A(z) = \prod_{k=1}^{K} \frac{1-a^*(k)z}{z-a(k)}$$
$$a(k) \in \{\mathbb{C}||a(k)| < 1\}, \quad \max_k\{|a(k)|\} < |z| \tag{3}$$

$$B(z) = \prod_{k=1}^{K-1} \frac{1-b^*(k)z}{z-b(k)}$$
$$b(k) \in \{\mathbb{C}||b(k)| < 1\}, \quad \max_k\{|b(k)|\} < |z| \tag{4}$$

with $*$ denoting the conjugate complex. The frequency responses of these allpass filters are written

$$A\left(e^{j\Omega}\right) = e^{-j\varphi_A(\Omega)} \tag{5a}$$

$$B\left(e^{j\Omega}\right) = e^{-j\varphi_B(\Omega)}. \tag{5b}$$

The allpass transformation of (2) is a bilinear transformation and converts (1) into the new transfer function

$$\tilde{H}_i(z) = B^{L-1}(z) \sum_{n=0}^{L-1} h(n) \cdot W_M^{-in} \cdot \left(\frac{A(z)}{B(z)}\right)^n \tag{6}$$
$$= \Psi(z) \sum_{n=0}^{L-1} h(n) \cdot W_M^{-in} \cdot \Theta^n(z)$$

$$\text{with } \Psi(z) = B^{L-1}(z) \quad \text{and} \quad \Theta(z) = \frac{A(z)}{B(z)} \tag{7}$$

to ease the notation. The common allpass transformation of first order is included as *special case* for $K=1$ so that

$$\Psi(z) = 1 \wedge \Theta(z) = \frac{1-a^*z}{z-a}. \tag{8}$$

The frequency responses of the uniform analysis subband filters given by (1) and the nonuniform analysis subband filters given by (6) are related by

$$\tilde{H}_i\left(e^{j\Omega}\right) = e^{-j(L-1)\varphi_B(\Omega)} \cdot H_i\left(e^{j\varphi_\Theta(\Omega)}\right) \tag{9}$$

$$\text{with } \varphi_\Theta(\Omega) = \varphi_A(\Omega) - \varphi_B(\Omega). \tag{10}$$

The phase difference of (10) ensures that the allpass transformation causes a *frequency warping* where a frequency interval of $\Delta\Omega = 2\pi$ is mapped onto an interval of 2π on the warped frequency scale:

$$[0,2\pi] \rightarrow [0,2\pi] : \Omega \mapsto \varphi_\Theta(\Omega). \tag{11}$$

In contrast, the allpass transformation $z^{-n} \rightarrow A^n(z)$ maps the frequency interval of $[0,2\pi]$ onto an interval of $\Delta\Omega = 2\pi K$ which causes an undesirable comb-filter effect for $K > 1$, cf., [11].

The warping characteristic is solely determined by $\varphi_\Theta(\Omega)$ and thus the transfer function $\Theta(z) = A(z)/B(z)$. However, dependent on the choice for $B(z)$, the transfer function $\Theta(z)$ can become either unstable or noncausal. The additional filter with transfer function $\Psi(z) = B^{L-1}(z)$ ensures that the warped subband filters of (6) are always stable and causal.

The function of (11) is *bijective*, if the continuous (unwrapped) phase response $\varphi_\Theta(\Omega)$ is monotone increasing, which is guaranteed by a positive group delay

$$\frac{\partial\varphi_\Theta(\Omega)}{\partial\Omega} > 0 \quad \forall\Omega. \tag{12}$$

This property is required to ensure a unique mapping so that a comb-filter effect is avoided. The choice

$$B(z) = z^{-(K-1)} \tag{13}$$

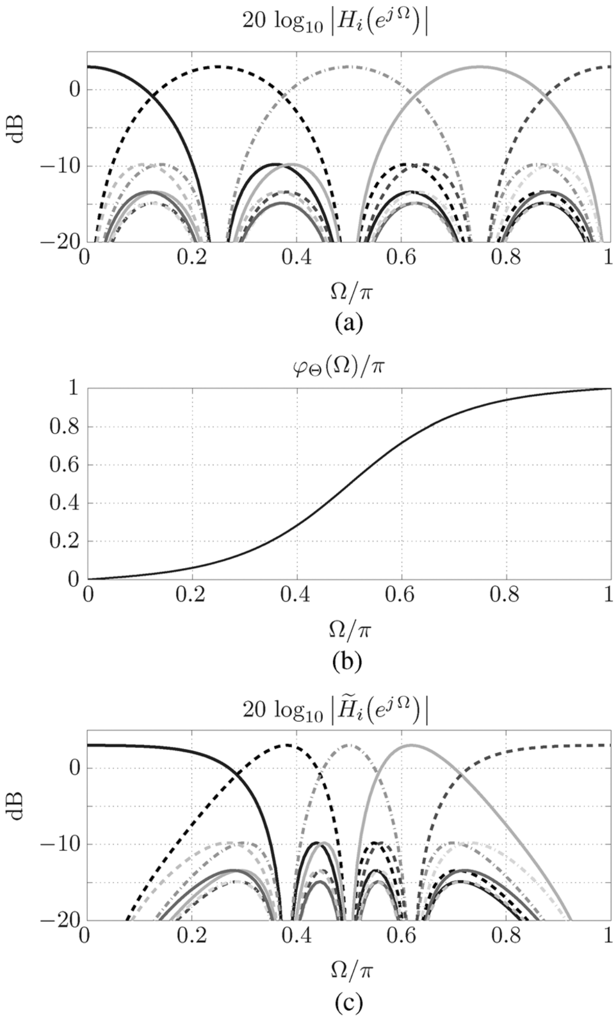

Fig. 1. DFT analysis filter bank designed by an allpass transformation of second order with $L = M = 8, a(1) = -j0.5, a(2) = j0.5, b(1) = 0$: (a) uniform analysis subband filters; (b) phase response of $\Theta(e^{j\Omega})$; and (c) warped analysis subband filters.

is of special interest as it reduces the implementation cost for the filter bank and simplifies the design procedure. With $a(k) = \alpha(k)e^{j\gamma(k)}$, the requirement of (12) can be written [11]

$$\sum_{k=1}^{K} \frac{1-\alpha^2(k)}{1-2\alpha(k)\cos(\Omega-\gamma(k))+\alpha^2(k)} > K-1 \quad \forall\Omega. \tag{14}$$

The effects of an allpass transformation of second order ($K = 2$) with complex allpass poles on a DFT analysis filter bank with $M = 8$ channels and rectangular prototype filter $h(n) = \sqrt{R}/M$ of length $L = M$ is illustrated in Fig. 1. It is easily verified (and visible) that the phase response $\varphi_\Theta(\Omega)$ fulfills (11) and (12). The bandwidths of the subband filters decrease first and increase afterwards within the interval $\Omega \in [0, \pi]$ since the phase response $\varphi_\Theta(\Omega)$ has one inflection point within this region. In contrast, such a flexible adjustment of the frequency resolution can not be achieved by an allpass transformation of first order.

An efficient implementation of the analysis filter bank can be obtained by rewriting (7) as follows:

$$\tilde{H}_i(z) = \Psi(z)\sum_{m=0}^{l_M-1}\sum_{\lambda=0}^{M-1} h(mM+\lambda)\cdot\Theta^{mM+\lambda}(z)\cdot W_M^{-\lambda i} \tag{15}$$

where it is assumed w.l.o.g. that $L = l_M M$ with $l_M \in \mathbb{N}$. Fig. 2 shows the corresponding polyphase network (PPN) implementation of the analysis filter bank, where the inverse discrete Fourier transform (IDFT) can be efficiently calculated by the inverse fast Fourier transform (IFFT).

The following *synthesis* subband filters are considered

$$\bar{G}_i(z) = \sum_{n=0}^{L-1} g(n)\cdot W_M^{-i(n+1)}\cdot P(z, L-1-n) \tag{16}$$

with $g(n)$ denoting the FIR of the synthesis prototype filter of length L. The N_p coefficients of the L transfer functions

$$P(z,n) = \sum_{\nu=0}^{N_p-1} p_n(\nu)\cdot z^{-\nu}, n \in \{0,1,\ldots,L-1\} \tag{17}$$

shall be determined in such a way that (almost) PR is achieved.

The FIR synthesis subband filters can be expressed by the PPN representation

$$\bar{G}_i(z) = \sum_{\lambda=0}^{M-1} \bar{G}^{(M)}_{M-1-\lambda}(z)\cdot W_M^{\lambda i}, \quad i \in \{0,1,\ldots,M-1\} \tag{18}$$

with the "modified" polyphase components

$$\bar{G}^{(M)}_{\lambda}(z) = \sum_{m=0}^{l_M-1} g(mM+\lambda)\cdot P(z,(l_M-m)M-1-\lambda). \tag{19}$$

This efficient PPN implementation of the synthesis filter bank is shown in Fig. 2. The *uniform* DFT AS FB is obtained for $\Psi(z) = 1, \Theta(z) = z^{-1}$ and $P(z,n) = z^{-(L-1-n)}$ with $n \in \{0,1,\ldots,L-1\}$.

III. Synthesis Filter-Bank Design

The output signal of the considered AS FB (shown in Fig. 2) can be written in the z-domain as follows:

$$Y(z) = \frac{1}{R}\sum_{r=0}^{R-1} X(zW_R^r)\sum_{i=0}^{M-1}\tilde{H}_i(zW_R^r)\cdot\bar{G}_i(z) \tag{20}$$

with $R \in \{\mathbb{N}|1 \le R \le M\}$. The AS FB with subsampling by R is a *linear periodically time-varying* (LPTV) system with period R since $W_R^r = W_R^{r+lR}$ for $l \in \mathbb{Z}$. This behavior is taken into account by determining the overall transfer function of the filter bank for R time-shifted unit sample sequences as input, i.e., $X(z) = z^{-l}$ for $l \in \{0,1,\ldots,R-1\}$. By this means, (20) turns into the new overall transfer function

$$T_l(z) = \frac{Y(z)}{z^{-l}} = \frac{1}{R}\sum_{r=0}^{R-1} W_R^{-rl}\sum_{i=0}^{M-1}\tilde{H}_i(zW_R^r)\cdot\bar{G}_i(z). \tag{21}$$

For our design, a matrix representation of the transfer function $T_l(z)$ in dependence of the unknown LN_p coefficients $p_n(\nu)$ of the synthesis filter bank is required. In the following, bold lower-case variables denote vectors and matrices are marked by bold upper-case variables. The superscripts T and H indicate the transpose and Hermitian transpose of a vector or matrix. The $L \times L$ identity matrix is denoted by $\boldsymbol{I}_L$ and $\otimes$ marks the Kronecker product of two matrices. Some manipulations of (16) lead to the representation

$$\bar{G}_i(z) = \boldsymbol{v}_i^T\cdot\boldsymbol{D}^T(z)\cdot\boldsymbol{p}, \quad i \in \{0,1,\ldots,M-1\} \tag{22a}$$

$$\text{with } \boldsymbol{v}_i = \Big[g(L-1), g(L-2)W_M^{-(L-1)i},\ldots$$

$$\ldots, g(1)W_M^{-2i}, g(0)W_M^{-i}\Big]^T \tag{22b}$$

$$\boldsymbol{D}(z) = \boldsymbol{I}_L \otimes \boldsymbol{d}_{N_p}(z) \tag{22c}$$

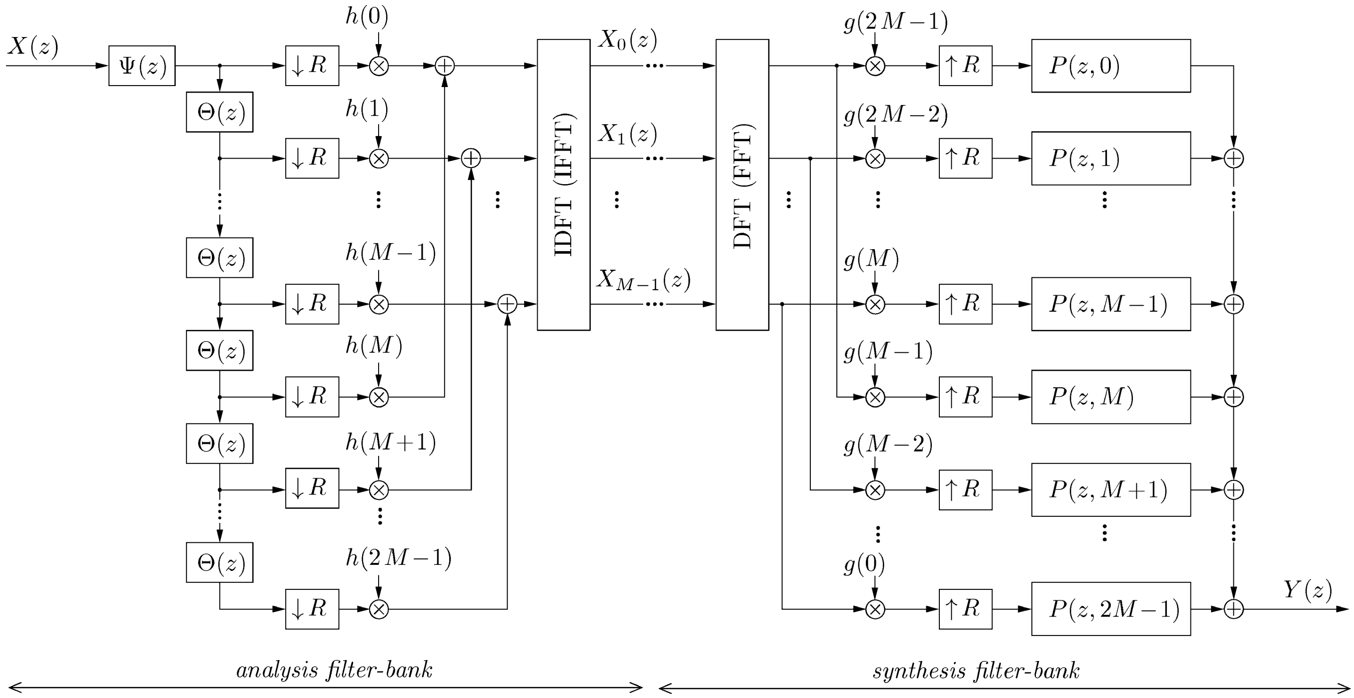

Fig. 2. PPN implementation of the allpass transformed DFT AS FB with downsampling by R and $L = 2M$ (without spectral processing).

$$\boldsymbol{d}_{N_p}(z) = \left[1, z^{-1}, \ldots, z^{-(N_p-1)}\right]^T \tag{22d}$$
$$\boldsymbol{p} = \left[\boldsymbol{q}_0^T, \boldsymbol{q}_1^T, \ldots, \boldsymbol{q}_{L-1}^T\right]^T \tag{22e}$$
$$\boldsymbol{q}_n = [p_n(0), p_n(1), \ldots, p_n(N_p-1)]^T \tag{22f}$$

for $n \in \{0, 1, \ldots, L-1\}$. With (22), the transfer function of (21) can now be formulated by the matrix notation

$$\begin{aligned} T_l(z) &= \left(\frac{1}{R}\sum_{r=0}^{R-1} W_R^{-rl} \sum_{i=0}^{M-1} \tilde{H}_i\left(zW_R^r\right) \cdot \boldsymbol{v}_i^T \cdot \boldsymbol{D}^T(z)\right) \cdot \boldsymbol{p} \\ &= \boldsymbol{\xi}(z,l) \cdot \boldsymbol{p}, \quad l \in \{0, 1, \ldots, R-1\} \end{aligned} \tag{23}$$

where the complex vector $\boldsymbol{\xi}(z,l)$ is of dimension $1 \times LN_p$.

A filter bank with *perfect reconstruction* is obtained, if the transfer function of (21) fulfills the condition

$$T_l(z) \overset{!}{=} z^{-d_0} \quad \forall l \in \{0, 1, \ldots, R-1\} \tag{24}$$

with d_0 marking the overall signal delay of the AS FB. This requirement can be expressed by means of (23):

$$\underbrace{\begin{bmatrix} \boldsymbol{\xi}(z,0) \\ \boldsymbol{\xi}(z,1) \\ \vdots \\ \boldsymbol{\xi}(z,R-1) \end{bmatrix}}_{=\boldsymbol{\Xi}_R(z)} \cdot \boldsymbol{p} \overset{!}{=} z^{-d_0} \cdot \mathbf{1}_R \tag{25}$$

where a column vector with R ones is denoted by $\mathbf{1}_R$. The LN_p coefficients of the vector $\boldsymbol{p}$ are determined by the requirement that (25) shall be fulfilled for $\mathcal{N} = LN_p$ discrete values on the unit circle

$$z = W_{\mathcal{N}}^n = e^{-j\frac{2\pi}{\mathcal{N}}n} \quad \forall n \in \{0, 1, \ldots, \mathcal{N}-1\}. \tag{26}$$

With the compact notation

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{\Xi}_R(1) \\ \boldsymbol{\Xi}_R(W_{\mathcal{N}}) \\ \vdots \\ \boldsymbol{\Xi}_R\left(W_{\mathcal{N}}^{\mathcal{N}-1}\right) \end{bmatrix} \in \mathbb{C}^{\mathcal{N}R \times \mathcal{N}} \tag{27a}$$

$$\boldsymbol{w}(d_0) = \begin{bmatrix} \mathbf{1}_R \\ W_{\mathcal{N}}^{-d_0} \cdot \mathbf{1}_R \\ \vdots \\ W_{\mathcal{N}}^{-(\mathcal{N}-1)d_0} \cdot \mathbf{1}_R \end{bmatrix} \in \mathbb{C}^{\mathcal{N}R \times 1} \tag{27b}$$

(25) turns into an R-times overdetermined set of $\mathcal{N}R$ *linear* equations

$$\boldsymbol{A} \cdot \boldsymbol{p} \overset{!}{=} \boldsymbol{w}(d_0). \tag{28}$$

A *least-squares error* solution is given by

$$\begin{aligned} \hat{\boldsymbol{p}} &= \arg\left\{\underset{\boldsymbol{p}}{\text{minimize}} \|\boldsymbol{A} \cdot \boldsymbol{p} - \boldsymbol{w}(d_0)\|_2\right\} \\ &= \boldsymbol{A}^{\#} \cdot \boldsymbol{w}(d_0) \end{aligned} \tag{29}$$

with $\boldsymbol{A}^{\#}$ denoting the pseudo-inverse of the matrix $\boldsymbol{A}$ provided that the matrix $\boldsymbol{A}^H\boldsymbol{A}$ is of full rank. Otherwise, a solution exists as well but is nonunique.

Some properties of this design approach are worth mentioning. The devised numerical LSE design strives towards a PR solution which can only be achieved with a certain accuracy. However, it will be shown later that these deviations are often such small that PR is actually achieved given the limited numerical precision to implement and evaluate a filter bank. A prerequisite for a PR solution is of course that the underlying uniform filter bank achieves already PR.

For many design examples, synthesis filters with a distinctive bandpass characteristic were obtained even though this is not an explicit design criterion. The incorporation of additional design constraints to improve such a behavior is also possible. However, this results in a more complex, constrained optimization and it remains to be verified whether the PR constraint can still be fulfilled. An advantage of the presented approach is still its simplicity as only a linear set of equations has to be solved according to (28).

Finding rules for the design parameters such that specific criteria are fulfilled is inherently more difficult for numerical designs than for analytical closed-form designs. For the signal delay, the choice $d_0 = N_p - 2R$ has been found suitable to achieve PR. For high subsampling rates, such as critical subsampling $(R = M)$, it is still possible to achieve PR, but the synthesis filters exhibit usually no distinctive bandpass characteristic even for a high filter length N_p. This might be tolerable for certain applications such as [13] where the proposed synthesis filters have a less pronounced bandpass characteristic. However, allpass transformed AS FBs are more commonly used for applications with noncritical subsampling such as speech enhancement systems, e.g., [4] and [5].

The obtained vector $\hat{\boldsymbol{p}}$ is often sparse with many coefficients (almost) equal to zero. For the same filter length N_p, the computational complexity of the new filter bank is comparable to that of [6], [9], and [10] as well as [12]–[14], where similar filter bank structures (but different designs) are proposed.

The properties of the new approach shall be examined by some design examples. First, the analysis filter bank of Fig. 1(c) is considered. The optimization of (29) is performed with parameters $N_p = 48, R = M/4 = 2$ and $d_0 = 44$. The obtained vector $\hat{\boldsymbol{p}}$ is rather sparse with about 48% of its coefficients having a value of less than $\pm 10^{-12}$. Fig. 3(a) shows that the obtained synthesis subband filters feature a bandpass characteristic which is very similar to that of the analysis subband filters plotted in Fig. 1(c). The frequency response of (21) $T_l(e^{j\Omega}) = |T_l(e^{j\Omega})|e^{-j\varphi_{T_l}(\Omega)}$ is evaluated for $l = 0$ by plotting its magnitude response $|T_0(e^{j\Omega})|$ and phase error $\Delta\varphi_T(\Omega) = \varphi_{T_0}(\Omega) - d_0\Omega$ in Fig. 3(b) and (c). These curves indicate actual a PR design.

A filter bank is a multirate system which can be analyzed by its system response $t_{\mathrm{bi}}(k_2, k_1)$, that is, the response of the system at instant k_2 to a unit sample sequence at k_1 [15]. The corresponding two-dimensional frequency-domain representation is given by the bifrequency system function $T_{\mathrm{bi}}(e^{j\Omega_2}, e^{j\Omega_1})$, which is plotted in Fig. 4 for the new design. It represents a PR system as no side diagonals with aliasing components occur and the main diagonal is a constant line with deviations equal to those of Fig. 3(b) (hence not plotted in greater detail).

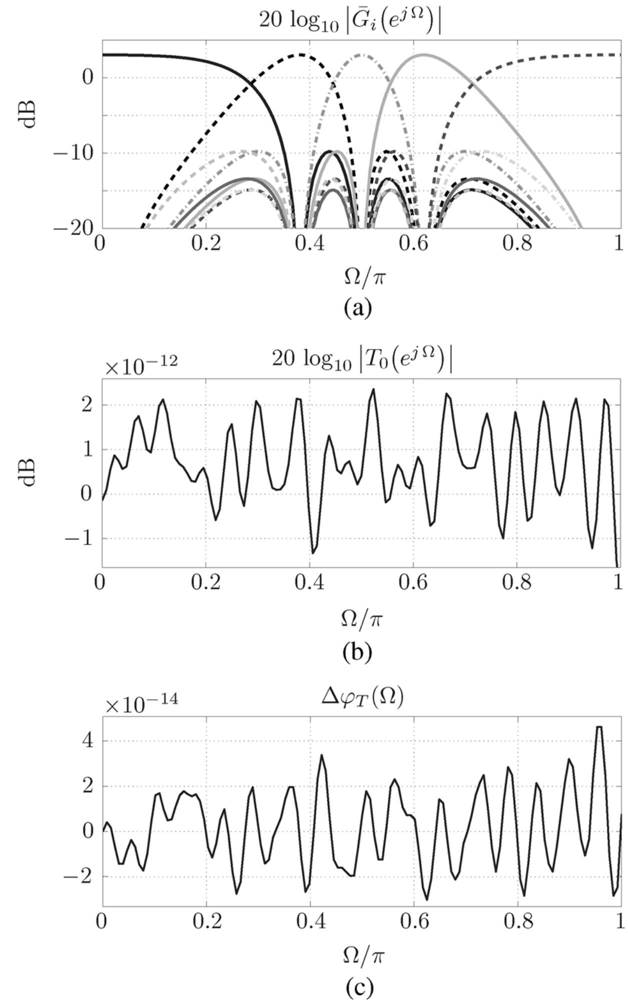

Fig. 3. Analysis of the new LSE synthesis filter bank design for $N_p = 48, R = M/4 = 2$ and $d_0 = 44$. (The analysis filter bank is shown in Fig. 1(c).): (a) synthesis subband filters; (b) overall transfer function; and (c) phase error.

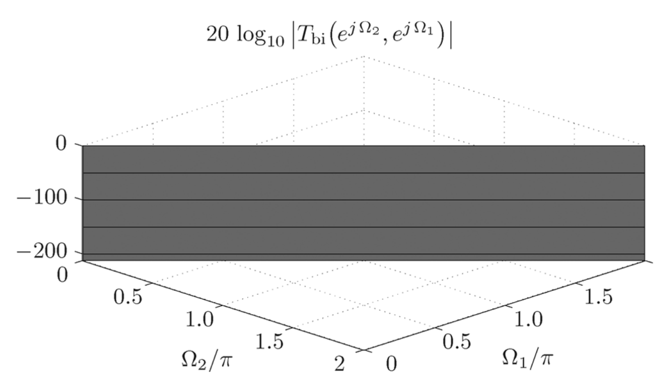

Fig. 4. Magnitude of bifrequency system function for LSE design of Fig. 3.

IV. Comparison With Closed-Form PR Designs

The proposed LSE filter bank design is now compared with the closed-form PR design of [14].[1] A DFT AS FB based on an allpass transformation of first order is considered with rectangular prototype filter and design parameters $a = 0.4$ and $R = (M/4) = 2$. In this case, the closed-form design provides a (stable) FIR synthesis filter bank with $N_p = 2R - 1$ and signal delay $d_0 = R - 1$. A comparison with the new LSE design is provided by Fig. 5. The new approach can achieve PR for the *same* parameters as the closed-form design.[2] In this case, LSE and closed-form design provide both synthesis filters with an insufficient bandpass characteristic. (This applies also for other parameters such as critical subsampling where $R = M$.) However, the new design enables the use of higher values for N_p and d_0 so that synthesis filters with a bandpass characteristic can be obtained while maintaining PR as shown by the solid curves in Fig. 5(a) and (b).

In contrast to the closed-form PR designs, the new design can also provide a PR solution for a filter bank with longer prototype filters where $L > M$. An example is given by the prototype filters

$$h(n) = g(n) = \frac{\sqrt{R}}{L}\left(1 - \sqrt{2}\cos\left(\frac{\pi}{M}n + \frac{1}{2}\right)\right) \tag{30}$$

[1]In [14], critical and noncritical subsampling is considered where the solution for critical subsampling is equivalent to that of [13].

[2]For all designs of this section, the phase error $\Delta\varphi_T(\Omega)/\pi$ were below $\pm 3 \cdot 10^{-14}$ and the bifrequency system function equal to that of Fig. 4.

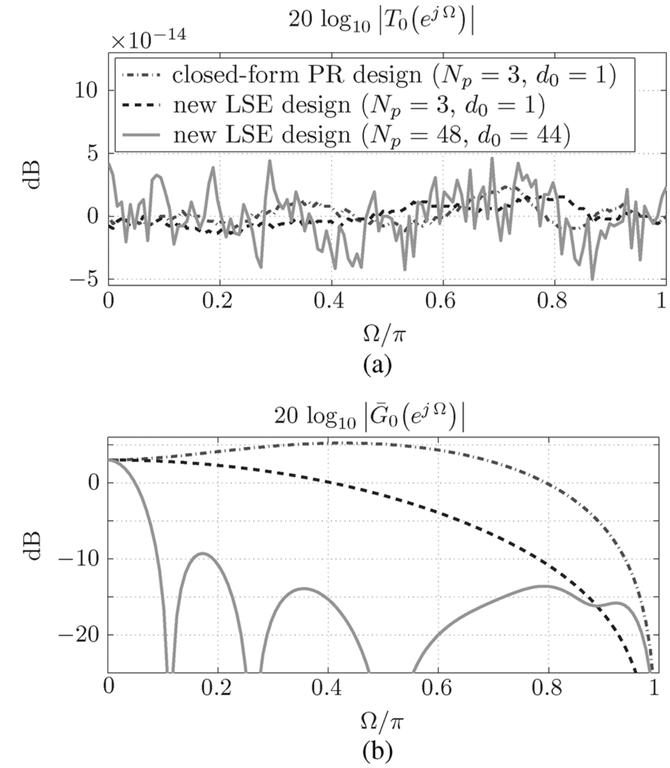

Fig. 5. Comparison of closed-form PR design and new LSE design for $K = 1, R = (M/4) = 2$ and $a = 0.4$: (a) overall transfer functions; (b) synthesis subband filters for $i = 0$.

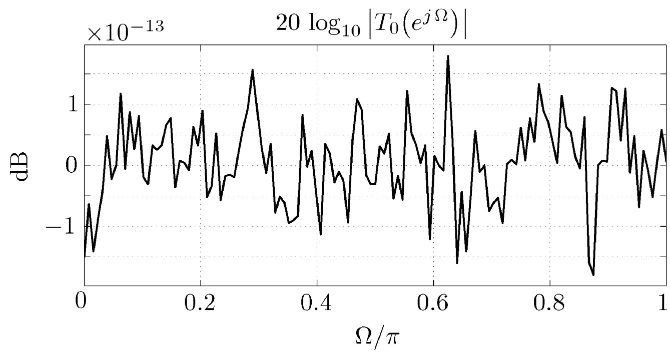

Fig. 6. Transfer function for LSE design of a warped DFT AS FB with parameters $K = 1, a = 0.4, L = 2M = 32, R = 2, N_p = 128, d_0 = 124$.

where $L = 2M$ and $n \in \{0, 1, \ldots, L-1\}$, cf., [9], [16]. The transfer function of a warped AS FB with such prototype filters and $M = 16$ channels is shown in Fig. 6. It can be observed that the LSE design yields again actually a PR system.

V. Conclusion

A new LSE design for a warped DFT AS FB with subsampling is presented whose time-frequency resolution is adjusted by an allpass transformation of first or higher order. The coefficients of the FIR synthesis filter bank are merely determined by a linear set of equations without constraints which can be easily solved. The synthesis subband filter are inherently causal and stable, and exhibit a significantly better bandpass characteristic than the closed-form PR filter bank designs of [12]–[14]. In addition, the signal delay is now an adjustable design parameter and the new approach provides also a solution for a PPN filter bank where the prototype filter length exceeds the number of channels $(L > M)$. For an appropriate choice of the design parameters, a filter bank with PR can be achieved effectively for an allpass transformation of first or higher order in contrast to the designs of [4]–[10]. A DFT filter bank is considered here, but the devised LSE design can also be applied to filter banks with other transformation kernels such as the discrete cosine transform (DCT).

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