Properties and Performance Bounds of Linear Analog Block Codes

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***Abstract*—Linear analog block codes have been considered for transmission of discrete-time and continuous-amplitude data. In this paper, the fidelity measure parameter SNR (pSNR) at the receiver is derived for an arbitrary generator matrix $\mathbf{P}$ using an additive white Gaussian noise (AWGN) channel. In contrast to [1], it is shown that the performance of linear analog bock codes is dependent on the eigenvalues of the matrix $\mathbf{P}^{\mathrm{T}}\mathbf{P}$ and not only on the dimensions of the matrix $\mathbf{P}$. Surprisingly, the quality of the received values is independent of the code rate r, and e.g. a simple identity matrix has the optimal eigenvalues. Furthermore, the theoretical fidelity bound OPTA (Optimum Performance Theoretically Attainable) is used to assess the performance of a transmission system of continuous-amplitude data.**

***Index Terms*—Linear Analog Block Codes, Analog Product Codes, OPTA**

I. Introduction

Analog block codes have been considered for discrete-time and continuous-amplitude data transmission. Continuous-amplitude data could be parameters of any speech, audio or video source codec. To protect the parameters against channel noise, analog coding measures can be applied. One possibility, the analog product codes, was presented e.g. in [2]. Redundancy is introduced by transmitting additional symbols which are the sums of the rows and columns of the data arranged to an $N \times N$ matrix. An iterative, turbo-like decoding algorithm was introduced in [2], which converges to the least squares (LS) solution. The optimal factor to control the convergence-speed was found in [1]. There, it was also derived that this type of analog product codes, which uses redundancy consisting of linear combinations of the data, can be reformulated as a linear analog block code with a generator matrix $\mathbf{P}$, whose LS-decoder uses just a multiplication with the pseudoinverse of $\mathbf{P}$ rather than performing iterative decoding.

In this contribution, we derive the optimal generator matrices in terms of pSNR. In Section II, different channel quality measures and theoretical fidelity bounds are shown. In Section III the properties of linear analog block codes are calculated. They are interpreted in Section IV.

II. Channel Quality and Bounds

A. $E_{\mathbf{u}}/N_0$ versus CSNR

Figure 1 shows a generic vector transmission system. The time index for successive data blocks is omitted for simplicity. The data vectors $\mathbf{u}$ are transformed to code vectors by a bijective function $f(\cdot)$. Noise $\mathbf{n}$ is added and an estimate $\hat{\mathbf{u}}$ is calculated by the function $g(\cdot)$. The zero mean data vectors consist of d_I components, each with a variance of σ_u^2. The zero mean code and noise vectors $\mathbf{y}$ and $\mathbf{n}$ consist of $d_C \geq d_I$ components with variances σ_n^2 and σ_y^2, respectively. The ratio of the dimensions is the code rate $r = d_I/d_C$. The energy of each vector is calculated by:

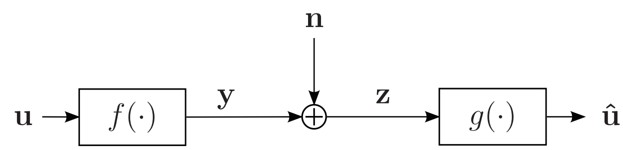

Fig. 1. A generic vector transmission system

$$E_{\mathbf{u}} = d_I \cdot \sigma_u^2, \tag{1}$$
$$E_{\mathbf{y}} = d_C \cdot \sigma_y^2, \tag{2}$$
$$E_{\mathbf{n}} = d_C \cdot \sigma_n^2. \tag{3}$$

Using the Nyquist theorem, the dimension of the vectors can also be interpreted as a factor proportional to the bandwidth (in Hz) used on the channel. To avoid sending more energy with decreasing code rates, the function $f(\cdot)$ comprises a normalization to yield $E_{\mathbf{y}} = E_{\mathbf{u}}$.

As a common measure for the channel quality we use the channel SNR (CSNR):

$$\mathrm{CSNR} = 10 \log_{10} \left(\frac{\sigma_y^2}{\sigma_n^2} \right). \tag{4}$$

Reformulating the fraction clarifies that the energy of the data vectors is normalized to the noise energy:

$$\frac{\sigma_y^2}{\sigma_n^2} = \frac{d_C E_{\mathbf{y}}}{d_C E_{\mathbf{n}}} = \frac{E_{\mathbf{y}}}{E_{\mathbf{n}}} = \frac{E_{\mathbf{u}}}{E_{\mathbf{n}}}. \tag{5}$$

Alternatively, we use a measure based on the noise power spectral density which is defined as

$$E_{\mathbf{u}}/N_0 = 10 \log_{10} \left(\frac{\sigma_u^2}{2\sigma_n^2} \right). \tag{6}$$

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Reformulating the fraction shows the normalization of the energy of the data vectors to the noise density

$$\frac{\sigma_u^2}{2\sigma_n^2} = \frac{E_{\mathbf{u}}}{2d_I\sigma_n^2}. \quad (7)$$

The measures are connected by:

$$\begin{aligned} \text{CSNR} &= 10\log_{10}\left(\frac{\sigma_y^2}{\sigma_n^2}\right) \\ &\overset{E_{\mathbf{u}}=E_{\mathbf{y}}}{=} E_{\mathbf{u}}/N_0 + 10\log_{10}(2r) \end{aligned} \quad (8)$$

When transmitting at code rates $r \neq 1$, these two measures are different. Without loss of generality, for different code rates a constant transmission time is assumed, resulting in the use of more bandwidth with a decreasing code rate. A constant CSNR implies a constant total noise energy on the channel, independent of the code rate and used bandwidth. In real transmission systems however, noise is present at all frequencies and is not reduced when the channel occupies a broader spectrum. A constant $E_{\mathbf{u}}/N_0$ implies a constant noise density, like in a physical channels. Therefore, the measure $E_{\mathbf{u}}/N_0$ permits a *fair* assessment between codes with different code rates whereas the CSNR measure does not.

B. Optimum Performance Theoretically Attainable (OPTA)

The goal in data transmission is to find a tradeoff between the minimization of transmission-time, energy and bandwidth consumption while obtaining a very low distortion at the receiver. For binary sources, the distortion is measured in terms of the bit error rate (BER). For continuous-amplitude sources, this measure is inapplicable, because a minimum distortion on the channel would still be rated as an erroneous transmission. Here, the mean square error (MSE) is a more appropriate measure ($d(u,\hat{u}) = \mathrm{E}\{(u-\hat{u})^2\}$). To obtain the fidelity of the whole transmission, the parameter SNR pSNR which is the MSE normalized to the variance of the source symbols is used:

$$\text{pSNR} = 10\log_{10}\left(\frac{\mathrm{E}\{u^2\}}{\mathrm{E}\{(u-\hat{u})^2\}}\right) \quad (9)$$

The minimum information rate that is necessary to describe source symbols u with a Gaussian distribution $p_{\mathcal{U}}(u) \sim N(0,\sigma_u^2)$ and a distortion D is given by the rate distortion function [3, (10.24)][1]:

$$R(D) = \begin{cases} \frac{1}{2}\,\mathrm{ld}\,\frac{\sigma_u^2}{D}, & 0 \le D \le \sigma_u^2 \\ 0 & D > \sigma_u^2. \end{cases} \quad (10)$$

The channel capacity, i.e., the maximum information rate that can be transmitted over a AWGN channel $\sim N(0,\sigma_n^2)$ is [3, (9.17)]:

$$C = \frac{1}{2}\,\mathrm{ld}\left(1+\frac{\sigma_y^2}{\sigma_n^2}\right) \text{ bits per use.} \quad (11)$$

The variances of the transmitted (coded) signal y and the noise n are σ_y^2 and σ_n^2, respectively. To achieve capacity, the information rate R has to match the channel capacity C. When transmitting d_I amplitude-continuous source symbols with the minimum information rate R according to (10) while using d_C times the channel with the capacity C, the following equation must hold:

$$R \cdot d_I = C \cdot d_C. \quad (12)$$

Evaluating (10) and (11) with (12) for $D \le \sigma_u^2$ and $r > 0$ results in

$$\begin{aligned} d_I \cdot \frac{1}{2}\,\mathrm{ld}\left(\frac{\sigma_u^2}{D}\right) &= d_C \cdot \frac{1}{2}\,\mathrm{ld}\left(1+\frac{\sigma_y^2}{\sigma_n^2}\right) \\ \Rightarrow \qquad \frac{\sigma_u^2}{D} &= \left(1+\frac{\sigma_y^2}{\sigma_n^2}\right)^{\frac{1}{r}}. \end{aligned} \quad (13)$$

The fraction $\frac{\sigma_y^2}{\sigma_n^2}$ corresponds to the channel SNR ($10^{\frac{\text{CSNR}}{10}}$) and $\frac{\sigma_u^2}{D}$ to the parameter SNR ($10^{\frac{\text{pSNR}}{10}}$):

$$\begin{aligned} \text{pSNR}^{\text{OPTA}} &= \frac{1}{r}\cdot 10\log_{10}\left(1+10^{\frac{\text{CSNR}}{10}}\right) \quad (14) \\ &= \frac{1}{r}\cdot 10\log_{10}\left(1+2r\cdot 10^{\frac{E_{\mathbf{u}}/N_0}{10}}\right) \quad (15) \end{aligned}$$

where (15) follows from (8). The maximum achievable pSNR for a given channel quality (14) and (15) is called OPTA (Optimum Performance Theoretically Attainable).

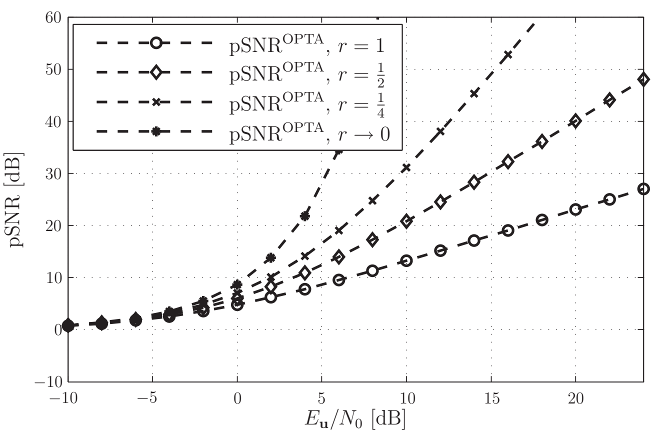

Fig. 2. OPTA with $E_{\mathbf{u}}/N_0$ as channel quality measure

Figure 2 shows OPTA for given code rates and channel qualities for Gaussian distributed source symbols.

III. Linear Analog Block Codes

Using linear analog block codes, the functions $f(\cdot)$ and $g(\cdot)$ in Fig. 1 are matrices such that encoding and decoding is performed by:

$$\mathbf{y} = \mathbf{P}\cdot\mathbf{u}, \quad (16)$$

$$\hat{\mathbf{u}} = \mathbf{P}^{+}\cdot\mathbf{z}. \quad (17)$$

The matrices $\mathbf{P}$ and $\mathbf{P}^{+}$ have the dimensions $d_C \times d_I$ and $d_I \times d_C$, respectively, and all vectors are column vectors.

[1] $\mathrm{ld}(\cdot)$ is the logarithm to the base 2

For calculating the performance of analog block coding using an arbitrary generator matrix $\mathbf{P}'$, this matrix has to be normalized by a factor α, i.e., $\mathbf{P} = \alpha\mathbf{P}'$. The normalization accounts for keeping the energy of the source and code vectors equal ($E_{\mathbf{y}} = E_{\mathbf{u}}$). The factor α is determined as follows:

$$\begin{aligned} E_{\mathbf{y}} &= \mathrm{E}\{||\mathbf{y}||^2\} = \mathrm{E}\{||\alpha\mathbf{P}'\mathbf{u}||^2\} = \alpha^2\,\mathrm{E}\{(\mathbf{P}'\mathbf{u})^{\mathrm{T}}(\mathbf{P}'\mathbf{u})\} \\ &= \alpha^2\,\mathrm{E}\{\mathbf{u}^{\mathrm{T}}\underbrace{\mathbf{P}'^{\mathrm{T}}\mathbf{P}'}_{\mathbf{G}':\,d_I\times d_I}\mathbf{u}\} = \alpha^2\,\mathrm{E}\left\{\sum_{k=1}^{d_I}\sum_{l=1}^{d_I} u_l g'_{l,k} u_k\right\} \\ &= \alpha^2 \sum_{k=1}^{d_I}\sum_{l=1}^{d_I} g'_{l,k}\,\mathrm{E}\{u_l u_k\} \stackrel{a)}{=} \alpha^2 \sum_{k=1}^{d_I}\sum_{l=1}^{d_I} g'_{l,k}\delta_{l,k} \\ &= \alpha^2 \sum_{k=1}^{d_I} g'_{l,l}\sigma_u^2 = \alpha^2\sigma_u^2 \cdot \mathrm{trace}\,(\mathbf{G}') \\ &= \alpha^2\sigma_u^2 \cdot \mathrm{trace}\left(\mathbf{P}'^{\mathrm{T}}\mathbf{P}'\right). \end{aligned} \tag{18}$$

Step $a)$ is due to $\mathrm{E}\{u_l u_k\} = \sigma_u^2 \cdot \mathbf{I}_{d_I}$, since the source symbols are statistically independent. Using (1) and $E_{\mathbf{y}} = E_{\mathbf{u}}$ results in

$$\begin{aligned} d_I \cdot \sigma_u^2 &= \alpha^2\sigma_u^2 \cdot \mathrm{trace}\left(\mathbf{P}'^{\mathrm{T}}\mathbf{P}'\right) \\ \Rightarrow \qquad \alpha &= \sqrt{\frac{d_I}{\mathrm{trace}\,(\mathbf{P}'^{\mathrm{T}}\mathbf{P}')}}. \end{aligned} \tag{19}$$

Therefore,

$$\mathbf{P} = \sqrt{\frac{d_I}{\mathrm{trace}\,(\mathbf{P}'^{\mathrm{T}}\mathbf{P}')}} \cdot \mathbf{P}' \tag{20}$$

can be used to normalize any given generator matrix $\mathbf{P}'$ to satisfy $E_{\mathbf{y}} = E_{\mathbf{u}}$.

The minimum variance unbiased estimator which has no a-priori information available is the maximum likelihood (ML) estimator [4]. For Gaussian noise, the least squares (LS) estimator yields the same results as the ML-estimator and can be derived by minimizing $||\mathbf{z} - \mathbf{P}\hat{\mathbf{u}}||^2$. This has already been shown in [1], but is repeated here for the sake of completeness:

$$\begin{aligned} ||\mathbf{z} - \mathbf{P}\hat{\mathbf{u}}||^2 &= (\mathbf{z} - \mathbf{P}\hat{\mathbf{u}})^{\mathrm{T}}(\mathbf{z} - \mathbf{P}\hat{\mathbf{u}}) \\ &= (\mathbf{z}^{\mathrm{T}} - \hat{\mathbf{u}}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}})(\mathbf{z} - \mathbf{P}\hat{\mathbf{u}}) \\ &= \mathbf{z}^{\mathrm{T}}\mathbf{z} - \mathbf{z}^{\mathrm{T}}\mathbf{P}\hat{\mathbf{u}} - \hat{\mathbf{u}}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}\mathbf{z} + \hat{\mathbf{u}}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}\mathbf{P}\hat{\mathbf{u}}. \end{aligned} \tag{21}$$

Differentiating the MSE w.r.t. $\hat{\mathbf{u}}$ and setting it to 0 yields for invertible $\mathbf{P}^{\mathrm{T}}\mathbf{P}$:

$$\frac{\partial||\mathbf{z} - \mathbf{P}\hat{\mathbf{u}}||^2}{\partial\hat{\mathbf{u}}} = 0$$

$$\Leftrightarrow \qquad -2\mathbf{P}^{\mathrm{T}}\mathbf{z} + 2\mathbf{P}^{\mathrm{T}}\mathbf{P}\hat{\mathbf{u}} = 0 \tag{22}$$

$$\Leftrightarrow \qquad \hat{\mathbf{u}} = (\mathbf{P}^{\mathrm{T}}\mathbf{P})^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{z} \tag{23}$$

Equation (22) follows from [5, (57) and (69)]. The expression $(\mathbf{P}^{\mathrm{T}}\mathbf{P})^{-1}\mathbf{P}^{\mathrm{T}}$ is the pseudoinverse of $\mathbf{P}$ and will be denoted $\mathbf{P}^{+}$. Thus, decoding with $\hat{\mathbf{u}} = \mathbf{P}^{+}\mathbf{z}$ yields the LS-optimal estimate. On the way to calculate the pSNR, first, the error and the variance of the error after decoding is considered:

$$\begin{aligned} \mathbf{u} - \hat{\mathbf{u}} &= \mathbf{u} - \mathbf{P}^{+}\mathbf{z} = \mathbf{u} - \mathbf{P}^{+}(\mathbf{P}\mathbf{u} + \mathbf{n}) \\ &= \mathbf{u} - \underbrace{(\mathbf{P}^{\mathrm{T}}\mathbf{P})^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{P}}_{\mathbf{I}_{d_I\times d_I}}\mathbf{u} + \mathbf{P}^{+}\mathbf{n} \\ &= \mathbf{P}^{+}\mathbf{n} \end{aligned} \tag{24}$$

$$\begin{aligned} \mathrm{E}\left\{||\mathbf{u} - \hat{\mathbf{u}}||^2\right\} &= \mathrm{E}\left\{||\mathbf{P}^{+}\mathbf{n}||^2\right\} \\ &\stackrel{b)}{=} \mathrm{trace}\left(\mathbf{P}^{+}\mathbf{M}\mathbf{P}^{+\mathrm{T}}\right) + (\mathbf{P}^{+}\mathbf{m})^{\mathrm{T}}\mathbf{P}^{+}\mathbf{m} \end{aligned}$$

Step $b)$ follows from [5, (252)] with $\mathbf{M} = \mathrm{E}\{\mathbf{n}^{\mathrm{T}}\mathbf{n}\}$ and $\mathbf{m} = \mathrm{E}\{\mathbf{n}\}$. Continuing:

$$\begin{aligned} \mathrm{E}\left\{||\mathbf{u} - \hat{\mathbf{u}}||^2\right\} &\stackrel{c)}{=} \sigma_n^2 \cdot \mathrm{trace}\left(\mathbf{P}^{+}\mathbf{P}^{+\mathrm{T}}\right) \\ &= \sigma_n^2 \cdot \mathrm{trace}\left((\mathbf{P}^{\mathrm{T}}\mathbf{P})^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{P}(\mathbf{P}^{\mathrm{T}}\mathbf{P})^{-1}\right) \\ &= \sigma_n^2 \cdot \mathrm{trace}\left((\mathbf{P}^{\mathrm{T}}\mathbf{P})^{-1}\right) \qquad (25) \\ &= \sigma_n^2 \cdot \sum_{i=1}^{d_I}\frac{1}{\gamma_i} \text{ with } \gamma_i = \mathrm{eig}(\mathbf{P}^{\mathrm{T}}\mathbf{P}, i). \qquad (26) \end{aligned}$$

Step $c)$ is due to the zero mean of the noise ($\mathbf{m} = \mathrm{E}\{\mathbf{n}\} = 0$) and due to the statistical independence of the noise $\mathbf{M} = \mathrm{E}\{\mathbf{n}^{\mathrm{T}}\mathbf{n}\} = \sigma_n^2 \cdot \mathbf{I}_{d_C}$. The derivation of (26) occupies several steps. Analogously to $\mathbf{G}'$ in (18), the matrix $\mathbf{G}$ is defined as $\mathbf{G} = \mathbf{P}^{\mathrm{T}}\mathbf{P}$. Since $\mathbf{G}$ is symmetric, i.e., $\mathbf{G} = \mathbf{G}^{\mathrm{T}}$, the eigenvalues of $\mathbf{G}$, $\mathrm{eig}(\mathbf{G}, i) = \gamma_i,\ i = 1 \ldots \mathrm{Rank}(\mathbf{G})$ have the following property [5, (215)]):

$$\mathrm{eig}(\mathbf{G}^{-1}, i) = \gamma_i^{-1} \tag{27}$$

With [5, (12)] follows

$$\mathrm{trace}(\mathbf{G}^{-1}) = \sum_{i=1}^{\mathrm{Rank}(\mathbf{G})} \gamma_i^{-1} \tag{28}$$

and therefore (26). Using (26) and (1), the parameter SNR for linear analog block codes $\mathrm{pSNR}^{\mathrm{LABC}}$ can be calculated:

$$\begin{aligned} \mathrm{pSNR}^{\mathrm{LABC}} &= 10\log_{10}\left(\frac{\mathrm{E}\{||\mathbf{u}||^2\}}{\mathrm{E}\{||\mathbf{u} - \hat{\mathbf{u}}||^2\}}\right) \\ &= 10\log_{10}\left(\frac{\sigma_u^2}{\sigma_n^2\frac{1}{d_I}\sum_{i=1}^{d_I}\frac{1}{\gamma_i}}\right), \gamma_i = \mathrm{eig}(\mathbf{P}^{\mathrm{T}}\mathbf{P}, i) \qquad (29) \\ &= E_{\mathbf{u}}/N_0 - 10\log_{10}\left(\frac{1}{d_I}\sum_{i=1}^{d_I}\frac{1}{\gamma_i}\right) + 10\log_{10}(2) \qquad (30) \\ &= \mathrm{CSNR} - 10\log_{10}\left(\frac{1}{d_I}\sum_{i=1}^{d_I}\frac{1}{\gamma_i}\right) - 10\log_{10}(r) \qquad (31) \end{aligned}$$

Equations (30) and (31) allow to calculate the pSNR of any given normalized matrix $\mathbf{P}$. To maximize the pSNR the matrix $\mathbf{P}$ has to fulfill some criteria. The matrix $\mathbf{P}$ has to have full rank and all eigenvalues of $\mathbf{P}^{\mathrm{T}}\mathbf{P}$ have to be maximized subject to the energy constraint ($E_{\mathbf{y}} = E_{\mathbf{u}}$).

To calculate the maximum eigenvalues, first, the sum of all eigenvalues of $\mathbf{P}^{\mathrm{T}}\mathbf{P}$ has to be determined:

$$\begin{aligned}\sum_{i=1}^{d_I}\gamma_i &= \sum_{i=1}^{d_I}\operatorname{eig}\left(\mathbf{P}^{\mathrm{T}}\mathbf{P}, i\right) = \operatorname{trace}\left(\mathbf{P}^{\mathrm{T}}\mathbf{P}\right)\\ &= \alpha^2 \operatorname{trace}\left(\mathbf{P}'^{\mathrm{T}}\mathbf{P}'\right)\\ &= \frac{d_I}{\operatorname{trace}\left(\mathbf{P}'^{\mathrm{T}}\mathbf{P}'\right)}\operatorname{trace}\left(\mathbf{P}'^{\mathrm{T}}\mathbf{P}'\right) = d_I. \end{aligned} \tag{32}$$

Furthermore, we need to show that all eigenvalues are positive. For a given real vector $\mathbf{x}$ of dimension d_I, $\mathbf{w} = \mathbf{P}\mathbf{x}$ is a real vector with dimension d_C. Using $\mathbf{G} = \mathbf{P}^{\mathrm{T}}\mathbf{P}$ and

$$\begin{aligned} & \mathbf{w}^{\mathrm{T}}\mathbf{w} \geq 0 \quad \text{because } w^2 \geq 0\\ \Leftrightarrow\quad & \mathbf{x}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}\mathbf{P}\mathbf{x} \geq 0\\ \Leftrightarrow\quad & \mathbf{x}^{\mathrm{T}}\mathbf{G}\mathbf{x} \geq 0\\ \overset{d)}{\Leftrightarrow}\quad & \mathbf{G}\text{ is positive semidefinite}\\ \overset{e)}{\Leftrightarrow}\quad & \operatorname{eig}\left(\mathbf{G}, i\right) \geq 0\ \forall i \in \{1, 2, \ldots, \operatorname{Rank}(\mathbf{G})\}\\ \Leftrightarrow\quad & \operatorname{eig}\left(\mathbf{P}^{\mathrm{T}}\mathbf{P}, i\right) \geq 0\\ \overset{f)}{\Leftrightarrow}\quad & \operatorname{eig}\left(\mathbf{P}^{\mathrm{T}}\mathbf{P}, i\right) > 0 \end{aligned} \tag{33}$$

all eigenvalues are positive. Steps $d)$ and $e)$ are shown in [5, (359) and (360)]. The case of disappearing eigenvalues can be omitted, because $\mathbf{G}$ has to have full rank, due to the necessary existence of the pseudo inverse $\mathbf{P}^{+}$. Therefore, identity $f)$ follows.

To maximize the pSNR in either (29), (30) or (31), the sum in the denominator has to be minimized. Using the Lagrange multiplier method with the constraint in (32) it follows:

$$L = \frac{1}{d_I}\sum_{i=1}^{d_I}\frac{1}{\gamma_i} - \lambda\left(\sum_{i=1}^{d_I}\gamma_i - d_I\right)$$

$$\frac{\partial L}{\partial \gamma_j} := 0$$

$$\Leftrightarrow\quad 0 = -\frac{1}{d_I\gamma_j^2} - \lambda\left(1 - 0\right)$$

$$\Leftrightarrow\quad \lambda = -\frac{1}{d_I\gamma_j^2} \tag{34}$$

$$\Leftrightarrow\quad \gamma_{j_{1,2}} = \pm\sqrt{-\frac{1}{d_I\lambda}}. \tag{35}$$

Inserting (35) into (32) we obtain:

$$\begin{aligned} & \sum_{i=1}^{d_I}\left(\pm\sqrt{-\frac{1}{d_I\lambda}}\right) = d_I\\ \Leftrightarrow\quad & d_I\cdot\left(\pm\sqrt{-\frac{1}{d_I\lambda}}\right) = d_I\\ \Leftrightarrow\quad & d_I\lambda = -1\\ \overset{(35)}{\Leftrightarrow}\quad & \gamma_{j_1} = 1 \ \vee\ \gamma_{j_2} = -1. \end{aligned} \tag{36}$$

The case $\gamma_{j_2} = -1$ can be omitted due to (33). Thus, the maximum pSNR is obtained if all eigenvalues of $\mathbf{P}^{\mathrm{T}}\mathbf{P}$ are equal, and due to the normalization in (20), are equal to one. The maximum pSNR is therefore:

$$\begin{aligned}\mathrm{pSNR}_{\mathrm{max}}^{\mathrm{LABC}} &= E_{\mathbf{u}}/N_0 + 10\log_{10}\left(\frac{2d_I}{d_I}\right)\\ &= E_{\mathbf{u}}/N_0 + 3.01\,\mathrm{dB}\end{aligned} \tag{37}$$

$$\mathrm{pSNR}_{\mathrm{max}}^{\mathrm{LABC}} = \mathrm{CSNR} + 10\log_{10}\left(\frac{1}{r}\right). \tag{38}$$

IV. Interpretation

Equation (30) and (31) show that using either the $E_{\mathbf{u}}/N_0$ or the CSNR as a channel quality measure, the performance of linear analog block codes depends on the eigenvalues of the matrix $\mathbf{P}^{\mathrm{T}}\mathbf{P}$, as opposed to the claim in [1]. Furthermore, using the *fair* measure $E_{\mathbf{u}}/N_0$ for the channel quality, the code rate does not have any influence on the pSNR. The code rate appears in (31) only due to the relationship between CSNR and $E_{\mathbf{u}}/N_0$ (8). The eigenvalues of $\mathbf{P}^{\mathrm{T}}\mathbf{P}$ have the main influence on the pSNR. In [1, (20)] a similar derivation was published, but the pSNR only depended on the code rate and the CSNR. The contents of the generator matrix $\mathbf{P}$ did not have any influence. There is a shortcoming in the approach in [1], as an independence of the contents of $\mathbf{P}$ is disproved here. The maximum pSNR is reached when all eigenvalues of $\mathbf{P}^{\mathrm{T}}\mathbf{P}$ are one. This holds, e.g. for the identity matrix, Hadamard matrices or any zero padded versions of such matrices.

Figure 3 shows the maximum pSNR for an optimal generator matrix $\mathbf{P}$. For a code rate of one, the linear analog block codes approaches OPTA for high values of $E_{\mathbf{u}}/N_0$, but for any smaller code rate, the distance to OPTA grows.

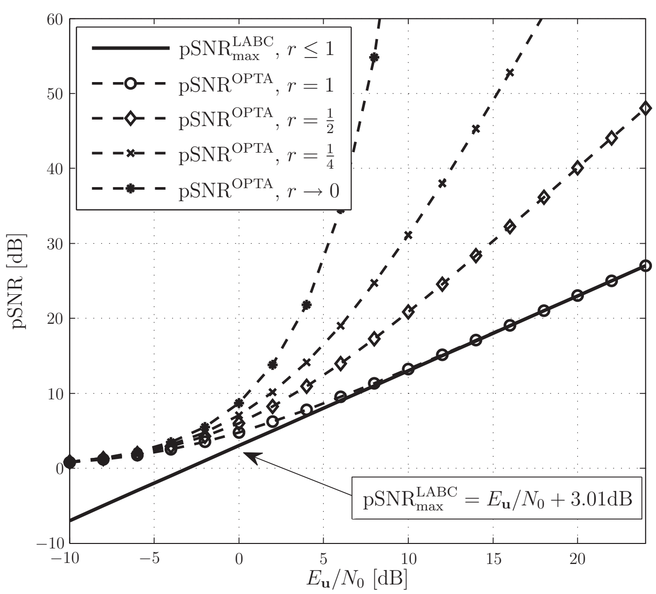

Fig. 3. OPTA and $\mathrm{pSNR}_{\mathrm{max}}$ over $E_{\mathbf{u}}/N_0$

Since one is the only code rate to approach OPTA and the identity matrix meets the requirement of equal and normalized eigenvalues, a simple scalar factor already achieves the best possible performance for linear analog block codes. Consequently, for AWGN channels, linear analog block codes with code rates smaller than one cannot be motivated.

V. SUMMARY

In this work, the performance of arbitrary linear analog block codes has been derived for either the CSNR or the $E_{\mathbf{u}}/N_0$ channel quality measure. The pSNR depends on $\mathrm{eig}(\mathbf{P}^{\mathrm{T}}\mathbf{P})$ and when using the $E_{\mathbf{u}}/N_0$ channel quality measure it is independent of the code rate. The dependency of the pSNR on the code rate for the channel quality measure CSNR is only due to the relationship between the CSNR and the $E_{\mathbf{u}}/N_0$ measure. Therefore, adding redundancy does not generate any improvement in pSNR at the receiver, instead it only wastes bandwidth and widens the gap to the theoretical upper bound OPTA. One ideal generator matrix $\mathbf{P}$ would be the identity matrix, which leads to the conclusion that for AWGN channels, linear analog block codes with code rates $r < 1$ are not reasonable.

REFERENCES

[1] O. Nemethova and M. Rupp, "Least-squares performance of analog product codes," *Conference Record of the Thirty-Eighth Asilomar Conference on Signals, Systems and Computers, 2004.*, vol. 2, pp. 2203–2207 Vol.2, Nov. 2004.

[2] M. Mura, W. Henkel, and L. Cottatellucci, "Iterative least-squares decoding of analog product codes," *IEEE International Symposium on Information Theory, 2003. ISIT 2003. Proceedings.*, p. 44, 2003.

[3] T.M. Cover and J.A. Thomas, *Elements of Information Theory*, Wiley-Interscience New York, 2006.

[4] S. M. Kay, *Fundamentals of statistical signal processing: estimation theory*, Prentice-Hall Signal Processing Series, 1993.

[5] K.B. Petersen and M.S. Pedersen, "The matrix cookbook," February 16, 2008.